Golden Prime Symmetry

Javier Grisales Herrera

University of Tolima

email: jgrisalesher@ut.edu.co

Ibagué, Colombia

June 27, 2025

Abstract

In this paper, the unpublished mathematical relation discovered by the author, about the existence of an equivalence relation on the infinite set of prime numbers, is demonstrated. That is, the golden ratio $(1+\sqrt{5})/2$ orders and classifies all prime numbers $p > 5$ into 8 infinite families according to their last and penultimate digit. That is, it induces a partition into 8 equivalence prime classes by 8 rational angles that are invariant under rotations of the regular pentagon in the complex plane. The prime classes in the complex plane correspond to the zeros of the cyclotomic polynomial number 20. Analogously, there is another equivalence relation on the infinite set of the twelfth Fibonacci numbers $F_{12(5m+j)}$, which has the same properties in the complex plane.

Key words: Prime number, golden ratio, twelfth Fibonacci, cyclotomic polynomial.

1 Introduction

To begin with, we will give the definitions of the golden ratio, the Fibonacci sequence, the notable golden angle, the Midy's theorem and its generalization. Then the definition of the *golden prime symmetry* and its theorems and propositions about the 8 equivalence classes discovered and proved by the author.

Also on a partition of the twelfth Fibonacci numbers into 8 equivalence classes corresponding to isometries of regular pentagons. This induces the existence of rotation matrices for the prime classes and for the twelfth Fibonacci classes.

To finally prove that the eigenvalues of the rotation matrices correspond to the 8 equivalence classes of the primes and the 8 equivalence classes of twelfth Fibonacci in the complex plane. They also correspond to the zeros of the cyclotomic polynomials 5, 10 and 20. Also, the prime equivalence classes module 20 correspond to the invertible elements in $\mathbb{Z}_{20}$.

Contents

2 Preliminaries

2.1 Golden ratio

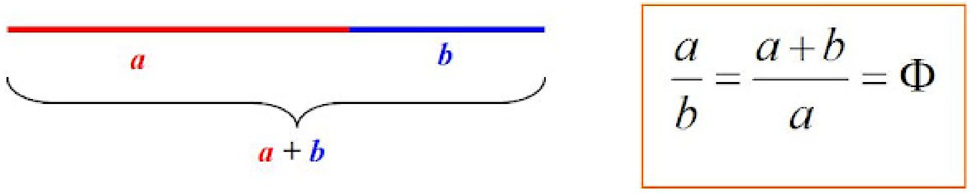

Figure 1: Major and minor segment.

Definition 3 of Book VI of Euclid's Elements (300 - 265 BC), defines the golden ratio as follows: *"A line is said to have been cut in extreme and half reason when the entire line is to the larger segment as the larger segment is to the smaller segment".*

Definition 2.1. Let a, b be two major and minor segments of a straight line $a + b$, we have [17]:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi = \frac{1+\sqrt{5}}{2} = 1,6180339887498948482045868343656... \quad (1)$$

Then,

$$\frac{a}{b} = \frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{1}{a/b}, \quad si \quad x = a/b. \quad (2)$$

As a result,

$$x = 1 + \frac{1}{x} \quad \Rightarrow \quad x^2 = x + 1 \quad \Rightarrow \quad x^2 - x - 1 = 0. \quad (3)$$

Resolving,

$$x_1 = \frac{1+\sqrt{5}}{2}; \qquad x_2 = \frac{1-\sqrt{5}}{2}. \quad (4)$$

2.2 Fibonacci sequence

The Fibonacci sequence was described in Europe by Leonardo of Pisa and published in his book "Liber Abaci" in 1202. It is defined as a recurrence sequence, where each term is obtained from the sum of the two previous terms [6,7].

$$F_n = F_{n-1} + F_{n-2}. \quad (5)$$

The golden ratio is the limit of the quotient of the nth Fibonacci term and its successor.

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\lim_{n\to\infty}\frac{F_n+F_{n-1}}{F_n}=\lim_{n\to\infty}1+\frac{F_{n-1}}{F_n}=1+\frac{1}{L} \tag{6}$$

$$L=1+\frac{1}{L} \quad\Rightarrow\quad L^2-L-1=0 \quad\Rightarrow\quad L_1=\frac{1+\sqrt{5}}{2};\quad L_2=\frac{1-\sqrt{5}}{2} \tag{7}$$

2.3 Notable golden angles

Definition 2.2. *We will call a notable golden angle an angle that is a multiple of* 18 *and that is not a multiple of* 5*, such that its image by means of the sine or cosine function is equal to a specific value of the golden ratio.*

These angles are grouped in two ways:

(i) *By multiplying* 18 *by the following odd elements* $\{1,3,7,9,11,13,17,19\}$*, we have:*

$$\sigma=\{18,54,126,162,198,234,306,342\}. \tag{8}$$

(ii) *By multiplying* 18 *by the following even elements* $\{2,4,6,8,12,14,16,18\}$*, we have:*

$$\delta=\{36,72,108,144,216,252,288.324\}. \tag{9}$$

Theorem 2.1. *The notable golden angles are of the form* $18(5m+j)$*, where* $m\in\mathbb{Z}$*,* $j\in 1,2,3,4$.

(1.) $18(5m+1)\equiv 18 \pmod{360} \quad\Leftrightarrow\quad m\equiv 0 \pmod 4$
(2.) $18(5m+1)\equiv 108 \pmod{360} \quad\Leftrightarrow\quad m\equiv 1 \pmod 4$
(3.) $18(5m+1)\equiv 198 \pmod{360} \quad\Leftrightarrow\quad m\equiv 2 \pmod 4$
(4.) $18(5m+1)\equiv 288 \pmod{360} \quad\Leftrightarrow\quad m\equiv 3 \pmod 4$

(5.) $18(5m+2)\equiv 36 \pmod{360} \quad\Leftrightarrow\quad m\equiv 0 \pmod 4$
(6.) $18(5m+2)\equiv 126 \pmod{360} \quad\Leftrightarrow\quad m\equiv 1 \pmod 4$
(7.) $18(5m+2)\equiv 216 \pmod{360} \quad\Leftrightarrow\quad m\equiv 2 \pmod 4$
(8.) $18(5m+2)\equiv 306 \pmod{360} \quad\Leftrightarrow\quad m\equiv 3 \pmod 4$

(9.) $18(5m+3)\equiv 54 \pmod{360} \quad\Leftrightarrow\quad m\equiv 0 \pmod 4$
(10.) $18(5m+3)\equiv 144 \pmod{360} \quad\Leftrightarrow\quad m\equiv 1 \pmod 4$
(11.) $18(5m+3)\equiv 234 \pmod{360} \quad\Leftrightarrow\quad m\equiv 2 \pmod 4$
(12.) $18(5m+3)\equiv 324 \pmod{360} \quad\Leftrightarrow\quad m\equiv 3 \pmod 4$

(13.) $18(5m+4)\equiv 72 \pmod{360} \quad\Leftrightarrow\quad m\equiv 0 \pmod 4$
(14.) $18(5m+4)\equiv 162 \pmod{360} \quad\Leftrightarrow\quad m\equiv 1 \pmod 4$
(15.) $18(5m+4)\equiv 252 \pmod{360} \quad\Leftrightarrow\quad m\equiv 2 \pmod 4$
(16.) $18(5m+4)\equiv 342 \pmod{360} \quad\Leftrightarrow\quad m\equiv 3 \pmod 4$

$18 = 18(1)$	$36 = 18(2)$
$54 = 18(3)$	$72 = 18(4)$
$126 = 18(7)$	$108 = 18(6)$
$162 = 18(9)$	$144 = 18(8)$
$198 = 18(11)$	$216 = 18(12)$
$234 = 18(13)$	$252 = 18(14)$
$306 = 18(17)$	$288 = 18(16)$
$342 = 18(19)$	$324 = 18(18)$

Table 1: Table of notable golden angles.

Degrees	Radians	Degrees	Radians
$18°$	$\pi/10$	$36°$	$\pi/5$
$54°$	$3\pi/10$	$72°$	$2\pi/5$
$126°$	$7\pi/10$	$108°$	$3\pi/5$
$162°$	$9\pi/10$	$144°$	$4\pi/5$
$198° \Leftrightarrow -162°$	$11\pi/10 \Leftrightarrow -9\pi/10$	$216° \Leftrightarrow -144°$	$6\pi/5 \Leftrightarrow -4\pi/5$
$234° \Leftrightarrow -126°$	$13\pi/10 \Leftrightarrow -7\pi/10$	$252° \Leftrightarrow -108°$	$7\pi/5 \Leftrightarrow -3\pi/5$
$306° \Leftrightarrow -54°$	$17\pi/10 \Leftrightarrow -3\pi/10$	$288° \Leftrightarrow -72°$	$8\pi/5 \Leftrightarrow -2\pi/5$
$342° \Leftrightarrow -18°$	$19\pi/10 \Leftrightarrow -\pi/10$	$324° \Leftrightarrow -36°$	$9\pi/5 \Leftrightarrow -\pi/5$

Table 2: Table of degrees and radians of notable golden angles.

This will further determine a partition of the prime numbers and another partition of the twelfth Fibonacci numbers. By multiplying 18 by the following elements multiples of 5 $\{5, 10, 15, 20\}$, is obtained $\{90, 180, 270, 360\}$, whose images under the sine and cosine function are $\{0, 1, -1\}$. That is, they do not map to golden ratio values.

In table 1, we can see the 16 notable golden angles. In table 2, we see their equivalences in radians. In table 3 and 4, we see their images under the sine and cosine function. We can consult the equivalences for the angles $18, 36, 54, 72$ in tables 3 and 4, on the website of Ron Knott, PhD from the University of Surrey: https://r-knott.surrey.ac.uk/fibonacci/simpletrig.html#section3.2.

2.4 Midy's theorem

Theorem 2.2. *Let a/p be a fraction, where $a < p$ and $p > 5$ is a prime number. Suppose further, that this fraction has a decimal period of even length, ie:*

$$\frac{a}{p} = 0.\overline{a_1 a_2 \ldots a_{2k-1} a_{2k}}$$

If we divide the period into two halves and add them together, we get a whole number consisting of only nines.

Proof.

Let us prove that the sum of the two halves of the period will always be equal to an integer consisting of nines, for a fraction a/p, with prime denominator, provided that the period has an even number of digits.

$2\sin(18)^\circ = \varphi^{-1}$	$2\cos(36)^\circ = \varphi$
$2\sin(54)^\circ = \varphi$	$2\cos(72)^\circ = \varphi^{-1}$
$2\sin(126)^\circ = \varphi$	$2\cos(108)^\circ = -\varphi^{-1}$
$2\sin(162)^\circ = \varphi^{-1}$	$2\cos(144)^\circ = -\varphi$
$2\sin(198)^\circ = -\varphi^{-1}$	$2\cos(216)^\circ = -\varphi$
$2\sin(234)^\circ = -\varphi$	$2\cos(252)^\circ = -\varphi^{-1}$
$2\sin(306)^\circ = -\varphi$	$2\cos(288)^\circ = \varphi^{-1}$
$2\sin(342)^\circ = -\varphi^{-1}$	$2\cos(324)^\circ = \varphi$

Table 3: Table of the 16 notable golden angles.

$2\sin(36)^\circ = \sqrt{3-\varphi}$	$2\cos(18)^\circ = \sqrt{2+\varphi}$
$2\sin(72)^\circ = \sqrt{2+\varphi}$	$2\cos(54)^\circ = \sqrt{3-\varphi}$
$2\sin(108)^\circ = \sqrt{2+\varphi}$	$2\cos(126)^\circ = -\sqrt{3-\varphi}$
$2\sin(144)^\circ = \sqrt{3-\varphi}$	$2\cos(162)^\circ = -\sqrt{2+\varphi}$
$2\sin(216)^\circ = -\sqrt{3-\varphi}$	$2\cos(198)^\circ = -\sqrt{2+\varphi}$
$2\sin(252)^\circ = -\sqrt{2+\varphi}$	$2\cos(234)^\circ = -\sqrt{3-\varphi}$
$2\sin(288)^\circ = -\sqrt{2+\varphi}$	$2\cos(306)^\circ = \sqrt{3-\varphi}$
$2\sin(324)^\circ = -\sqrt{3-\varphi}$	$2\cos(342)^\circ = \sqrt{2+\varphi}$

Table 4: Table of the 16 opposite notable golden angles.

If the length λ of the period is even, we can write $\lambda = 2l$. Moreover, if the two halves of the period P are A and B, then P is a number of l digits, while A and B have l digits each [10, 11, 12, 13, 19, 23].

Thus, we have:

$$P = A\ 10^l + B$$

The fraction $\frac{a}{p}$, can be expressed as:

$$\frac{a}{p} = \frac{P}{10^\lambda - 1} = \frac{A\ 10^l + B}{10^\lambda - 1} \tag{10}$$

Since $\lambda = 2l$, we have:

$$10^\lambda - 1 = 10^{2l} - 1 = (10^l - 1)(10^l + 1) \tag{11}$$

Since the fraction a/p is irreducible. This means that p divides $10^\lambda - 1$. If p divides to $10^\lambda - 1 = 10^{2l} - 1 = (10^l - 1)(10^l + 1)$, must divide at least one of the two factors, because p is prime. Now p cannot divide to $10^l - 1$, because $l < \lambda$ y λ is the smallest number for which $10^\lambda - 1$ is divisible by p. Therefore, p divides to $10^l + 1$. From (10) and (11), we have:

$$\frac{a}{p} = \frac{A\ 10^l + B}{(10^l - 1)(10^l + 1)}$$

Rewriting,

$$\frac{a(10^l+1)}{p} = \frac{A\ 10^l + B}{10^l - 1}$$

The left side is an integer, because p divides $10^l + 1$. Therefore, the right-hand side is also an integer number. Then,

$$\frac{a(10^l+1)}{p} = \frac{A\ 10^l + B}{10^l - 1}$$

$$\frac{a(10^l+1)}{p} = \frac{A\ 10^l - A + A + B}{10^l - 1}$$

$$\frac{a(10^l+1)}{p} = \frac{A\ (10^l - 1) + A + B}{10^l - 1}$$

$$\frac{a(10^l+1)}{p} = A + \frac{A + B}{10^l - 1}$$

Since A is an integer number, then

$$\frac{A+B}{10^l - 1} = h \tag{12}$$

Now A consists of l digits and is the largest when those digits are nines. The number formed by l digits 9 is $10^l - 1$. And therefore, we have to $A \leq 10^l - 1$ y $B \leq 10^l - 1$, so that:

$$A + B \leq 2\ (10^l - 1)$$

If $A+B = 2\ (10^l - 1)$, this implies that A and B have the value $10^l - 1$. But this would indicate that the period is not of even length, it would be of length one and consists of a single digit 9. So, the inequality is strict.

$$A + B < 2\ (10^l - 1) \tag{13}$$

Equation (12), can be written as:

$$A + B = h\ (10^l - 1)$$

where h is an integer. According to (13), $h < 2$, then, it must be 1. Then,

$$A + B = 10^l - 1$$

Therefore, $A + B$ is the number of l digits 99...99. $\square$

Example:

$$1/7 = 0.\overline{142857}.$$
$$1/11 = 0.\overline{09}.$$
$$1/13 = 0.\overline{076923}.$$
$$1/17 = 0.\overline{0588235294117647}.$$
$$1/19 = 0.\overline{052631578947368421}.$$
$$1/23 = 0.\overline{0434782608695652173913}.$$
$$1/29 = 0.\overline{0344827586206896551724137931}.$$

Let's look at the chains of nines:

$$1/7 : 142 + 857 = 999.$$
$$1/13 : 076 + 923 = 999.$$
$$1/17 : 05882352 + 94117647 = 99999999.$$
$$1/19 : 052631578 + 947368421 = 999999999.$$
$$1/23 : 04347826086 + 95652173913 = 99999999999.$$
$$1/29 : 03448275862068 + 96551724137931 = 99999999999999.$$

2.5 Generalization of Midy's theorem

In January of 2004, Brian Ginsberg, a student at Yale University generalized Midy's theorem to decimal expansions with period length $3n$. In the year 2005, Ankit Gupta and B. Sury generalized Midy's theorem to any number of blocks, [20], thus fully solving the case of $1/p$.

Example:

(1) The fraction 1/31, has a decimal period of length 15, that is, $3n$.

$$1/31 = 0.\overline{032258064516129}.$$
$$1/31 : 03225 + 80645 + 16129 = 99999.$$

(2) The fraction 1/127 is a 7-Midy fraction, that is, it can be divided into 7 blocks, such that:

$$\frac{1}{127} = 0.\overline{007874015748031496062992125984251968503937}.$$
$$007874 + 015748 + 031496 + 062992 + 125984 + 251968 + 503937 = 999999.$$

As we can see the length of the string of nines is equal to the length of each block.

(3) Even 2-Midy fractions can be expressed as 3-Midy fractions, if the period is of length $6n$.

The fraction 1/19, has a decimal period of length 18. Then, it can be divided into 3 blocks of 6 digits.

$$1/19 = 0.\overline{052631578947368421}.$$

$$052631 + 578947 + 368421 = 999999.$$

3 Golden Prime Symmetry

Based on what has already been described in the prelims, the author developed the following results. Starting from Midy's theorem and the notable golden angles, the golden prime symmetry is derived.

Definition 3.1. The *golden prime symmetry* is the analytical and algebraic relationship between the golden ratio and the prime numbers, by means of certain trigonometric functions in the real and complex plane in the form of equivalence classes corresponding to the zeros of certain cyclotomic polynomials.

Result: The golden ratio is the image of the discrete space of all prime numbers greater than 5 by a certain function. And the module 360 of the decimal period of the reciprocals of prime numbers induces a partition into 8 infinite equivalence classes which are invariant under the isometries of the regular pentagon and its opposite rotation.

4 Prime equivalence classes

Penult-odd primes

They are those primes whose penultimate digit is odd.

Proposition 1. The penult-odd primes are those primes congruent with $11, 13, 17, 19$ module 20.

$$\begin{aligned}
P_{1odd} &= \{p \in \mathbb{P} : \ p \equiv 11 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 18 \pmod{360}\big\}. \\
P_{3odd} &= \{p \in \mathbb{P} : \ p \equiv 13 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 126 \pmod{360}\big\}. \\
P_{7odd} &= \{p \in \mathbb{P} : \ p \equiv 17 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 54 \pmod{360}\big\}. \\
P_{9odd} &= \{p \in \mathbb{P} : \ p \equiv 19 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 162 \pmod{360}\big\}.
\end{aligned}$$

Penult-even primes

They are those primes whose penultimate digit is even.

Proposition 2. The penult-even primes are those primes congruent with $1, 3, 7, 9$ module 20.

$$\begin{aligned}
P_{1even} &= \{p \in \mathbb{P} : \ p \equiv 1 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 198 \pmod{360}\big\}. \\
P_{3even} &= \{p \in \mathbb{P} : \ p \equiv 3 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 306 \pmod{360}\big\}. \\
P_{7even} &= \{p \in \mathbb{P} : \ p \equiv 7 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 234 \pmod{360}\big\}. \\
P_{9even} &= \{p \in \mathbb{P} : \ p \equiv 9 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 342 \pmod{360}\big\}.
\end{aligned}$$

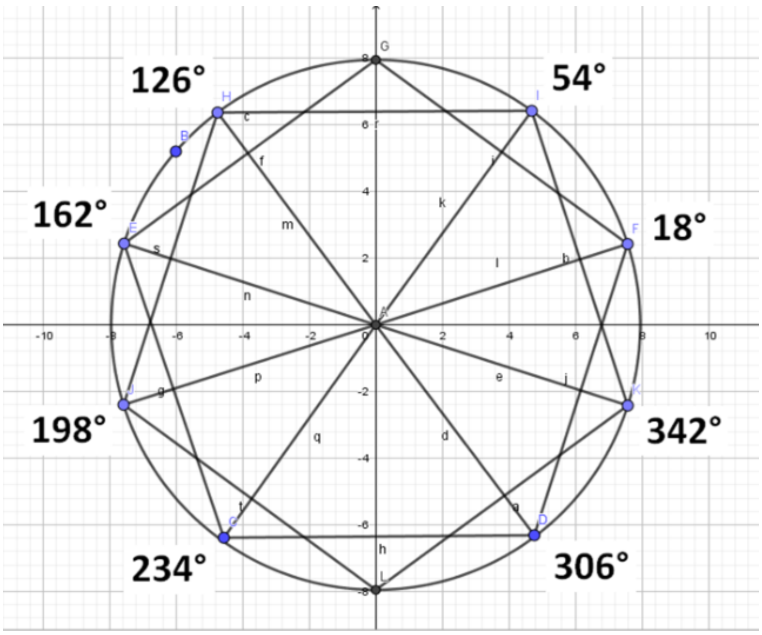

Figure 2: Prime classes at the vertices of two opposite pentagons.

Proof.

Let us now prove the propositions 1 and 2.

(1) **Prime class** P_{1odd}

$$P_{1odd} = \{p \in \mathbb{P} : \ p \equiv 11 \pmod{20}\} = \big\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 18 \pmod{360}\big\}.$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1} - 1) \equiv 18 \pmod{360}.$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20}.$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 11 \pmod{20} \;\Rightarrow\; \frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20}.$$

Suppose $p \equiv 11 \pmod{20}$, i.e., $p = 20k + 11$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+10}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+10} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k+10} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 11 \equiv 11 \pmod{20}.$$

The multiplicative inverse of 11 in $\mathbb{Z}_{20}$ is a number x, such that:

$$11x \equiv 1 \pmod{20}.$$

We have to,

$$11(11) = 121 \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} \equiv 11^{-1} \equiv 11 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1}-1) \equiv 11(19) = 209 \equiv 9 \pmod{20}.$$

We need to calculate $1/9$ module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by $1/9$, we have to:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 9(9) \equiv 1 \pmod{20}.$$

Then,

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20}.$$

Therefore,

$$p \equiv 11 \pmod{20} \;\Rightarrow\; \frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20}. \tag{14}$$

$\square$

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20} \Rightarrow\ p \equiv 11 \pmod{20}$$

Suppose,

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1}-1) \equiv 9 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1}-1 \equiv 9p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 9p+1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k+11$, for any integer k, is a congruence solution.

$$\begin{gathered} 10^{p-1} = 10^{20k+10} \equiv 9(20k+11)+1 \pmod{20}. \\ 0 \equiv 180k+99+1 \pmod{20}. \\ 0 \equiv 0 \pmod{20}. \end{gathered}$$

This confirms that $p = 20k+11$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{1odd} = \{p \in \mathbb{P} :\ \ p \equiv 11 \pmod{20}\} = \left\{p \in \mathbb{P} :\ \ 2p^{-1}(10^{p-1}-1) \equiv 18 \pmod{360}\right\}. \tag{15}$$

$\square$

(2) **Prime class** P_{3odd}.

$$P_{3odd} = \{p \in \mathbb{P} :\ \ p \equiv 13 \pmod{20}\} = \left\{p \in \mathbb{P} :\ \ 2p^{-1}(10^{p-1}-1) \equiv 126 \pmod{360}\right\}.$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1}-1) \equiv 126 \pmod{360}.$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 7 \pmod{20}.$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 13 \pmod{20}\ \Rightarrow\ \frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 7 \pmod{20}$$

Suppose $p \equiv 13 \pmod{20}$, i.e., $p = 20k + 13$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+12}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+12} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k+12} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 13 \equiv 13 \pmod{20}.$$

The multiplicative inverse of 13 in $\mathbb{Z}_{20}$ is a number x, such that:

$$13x \equiv 1 \pmod{20}.$$

We have to,

$$13(17) = 221 \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} \equiv 13^{-1} \equiv 17 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1} - 1) \equiv 17(19) = 323 \equiv 3 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9(3) \equiv 7 \pmod{20}.$$

Then,

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 7 \pmod{20}.$$

Therefore,

$$p \equiv 13 \pmod{20} \Rightarrow \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 7 \pmod{20}. \tag{16}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 7 \pmod{20} \Rightarrow p \equiv 13 \pmod{20}$$

Suppose,

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 7 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1}-1) \equiv 63 \equiv 3 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1}-1 \equiv 3p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 3p+1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k+13$, for any integer k, is a congruence solution.

$$\begin{gathered} 10^{p-1} = 10^{20k+12} \equiv 3(20k+13)+1 \pmod{20}. \\ 0 \equiv 60k+39+1 \pmod{20}. \\ 0 \equiv 0 \pmod{20}. \end{gathered}$$

This confirms that $p = 20k+13$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{3odd} = \{p \in \mathbb{P} : \quad p \equiv 13 \pmod{20}\} = \left\{p \in \mathbb{P} : \quad 2p^{-1}(10^{p-1}-1) \equiv 126 \pmod{360}\right\}. \tag{17}$$

□

(3) **Prime class** P_{7odd}.

$$P_{7odd} = \{p \in \mathbb{P} : \quad p \equiv 17 \pmod{20}\} = \left\{p \in \mathbb{P} : \quad 2p^{-1}(10^{p-1}-1) \equiv 54 \pmod{360}\right\}$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1}-1) \equiv 54 \pmod{360}$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 3 \pmod{20}$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 17 \pmod{20} \;\Rightarrow\; \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 3 \pmod{20}$$

Suppose $p \equiv 17 \pmod{20}$, i.e., $p = 20k+17$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+16}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+16} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1}-1 = 10^{20k+16}-1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 17 \equiv 17 \pmod{20}.$$

The multiplicative inverse of 17 in $\mathbb{Z}_{20}$ is a number x, such that:

$$17x \equiv 1 \pmod{20}.$$

We have to,

$$17(13) = 221 \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} \equiv 17^{-1} \equiv 13 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1} - 1) \equiv 13(19) = 247 \equiv 7 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9(7) = 63 \equiv 3 \pmod{20}.$$

Then,

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 3 \pmod{20}.$$

Therefore,

$$p \equiv 17 \pmod{20} \;\Rightarrow\; \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 3 \pmod{20}. \tag{18}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 3 \pmod{20} \Rightarrow p \equiv 17 \pmod{20}$$

Suppose,

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 3 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1} - 1) \equiv 27 \equiv 7 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1} - 1 \equiv 7p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 7p + 1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k + 17$, for any integer k, is a congruence solution.

$$10^{p-1} = 10^{20k+16} \equiv 7(20k + 17) + 1 \pmod{20}.$$
$$0 \equiv 140k + 119 + 1 \pmod{20}.$$
$$0 \equiv 0 \pmod{20}.$$

This confirms that $p = 20k + 17$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{7odd} = \{p \in \mathbb{P} : \ p \equiv 17 \pmod{20}\} = \left\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 54 \pmod{360}\right\}. \qquad (19)$$

□

(4) **Prime class** P_{9odd}.

$$P_{9odd} = \{p \in \mathbb{P} : \ p \equiv 19 \pmod{20}\} = \left\{p \in \mathbb{P} : \ 2p^{-1}(10^{p-1} - 1) \equiv 162 \pmod{360}\right\}$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1} - 1) \equiv 162 \pmod{360}$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20}$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 19 \pmod{20} \ \Rightarrow \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20}$$

Suppose $p \equiv 19 \pmod{20}$, i.e., $p = 20k + 19$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+18}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+18} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k+18} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 19 \equiv 19 \pmod{20}.$$

The multiplicative inverse of 19 in $\mathbb{Z}_{20}$ is a number x, such that:

$$19x \equiv 1 \pmod{20}.$$

We have to,

$$19(19) = 361 \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} \equiv 19^{-1} \equiv 19 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1} - 1) \equiv 19(19) = 361 \equiv 1 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9(1) \equiv 9 \pmod{20}.$$

Then,

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20}.$$

Therefore,

$$p \equiv 19 \pmod{20} \;\Rightarrow\; \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20}. \tag{20}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20} \Rightarrow\; p \equiv 19 \pmod{20}$$

Suppose,

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1} - 1) \equiv 81 \equiv 1 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1} - 1 \equiv p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv p + 1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k + 19$, for any integer k, is a congruence solution.

$$10^{p-1} = 10^{20k+18} \equiv (20k+19)+1 \pmod{20}.$$
$$0 \equiv 20k+19+1 \pmod{20}.$$
$$0 \equiv 0 \pmod{20}.$$

This confirms that $p = 20k+19$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{9odd} = \{p \in \mathbb{P}: \quad p \equiv 19 \pmod{20}\} = \left\{p \in \mathbb{P}: \quad 2p^{-1}(10^{p-1}-1) \equiv 162 \pmod{360}\right\}. \quad (21)$$

□

(5) **Prime class** P_{1even}.

$$P_{1even} = \{p \in \mathbb{P}: \quad p \equiv 1 \pmod{20}\} = \left\{p \in \mathbb{P}: \quad 2p^{-1}(10^{p-1}-1) \equiv 198 \pmod{360}\right\}$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1}-1) \equiv 198 \pmod{360}$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20}$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 1 \pmod{20} \;\Rightarrow\; \frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20}$$

Suppose $p \equiv 1 \pmod{20}$, i.e., $p = 20k+1$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k+1 \equiv 1 \pmod{20}.$$

The multiplicative inverse of 1 en $\mathbb{Z}_{20}$ is a number x, such that:

$$1x \equiv 1 \pmod{20}.$$

We have to,

$$1(1) \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} = 1^{-1} \equiv 1 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1}-1) = 1(19) \equiv 19 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 9(19) = 171 \equiv 11 \pmod{20}.$$

Then,

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20}.$$

Therefore,

$$p \equiv 1 \pmod{20} \;\Rightarrow\; \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20}. \tag{22}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20} \Rightarrow\; p \equiv 1 \pmod{20}$$

Suppose,

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1}-1) \equiv 99 \equiv 19 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1}-1 \equiv 19p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 19p+1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k+1$, for any integer k, is a congruence solution.

$$\begin{gathered} 10^{p-1} = 10^{20k} \equiv 19(20k+1)+1 \pmod{20}. \\ 0 \equiv 380k+19+1 \pmod{20}. \\ 0 \equiv 0 \pmod{20}. \end{gathered}$$

This confirms that $p = 20k+1$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{1even} = \{p \in \mathbb{P} : \;\; p \equiv 1 \pmod{20}\} = \left\{p \in \mathbb{P} : \;\; 2p^{-1}(10^{p-1} - 1) \equiv 198 \pmod{360}\right\}. \qquad (23)$$

□

(6) **Prime class** P_{3even}.

$$P_{3even} = \{p \in \mathbb{P} : \;\; p \equiv 3 \pmod{20}\} = \left\{p \in \mathbb{P} : \;\; 2p^{-1}(10^{p-1} - 1) \equiv 306 \pmod{360}\right\}$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1} - 1) \equiv 306 \pmod{360}$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 17 \pmod{20}$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 3 \pmod{20} \;\Rightarrow\; \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 17 \pmod{20}$$

Suppose $p \equiv 3 \pmod{20}$, i.e., $p = 20k + 3$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+2}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+2} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k+2} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 3 \equiv 3 \pmod{20}.$$

The multiplicative inverse of 3 in $\mathbb{Z}_{20}$ is a number x, such that:

$$3x \equiv 1 \pmod{20}.$$

We have to,

$$3(7) \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} = 3^{-1} \equiv 7 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1} - 1) = 7(19) \equiv 133 \equiv 13 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 9(13) = 117 \equiv 17 \pmod{20}.$$

Then,

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 17 \pmod{20}.$$

Therefore,

$$p \equiv 3 \pmod{20} \;\Rightarrow\; \frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 17 \pmod{20}. \tag{24}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 17 \pmod{20} \Rightarrow\; p \equiv 3 \pmod{20}$$

Suppose,

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 17 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1}-1) \equiv 153 \equiv 13 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1}-1 \equiv 13p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 13p+1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k+3$, for any integer k, is a congruence solution.

$$\begin{gathered} 10^{p-1} = 10^{20k+2} \equiv 13(20k+3)+1 \pmod{20}. \\ 0 \equiv 260k+39+1 \pmod{20}. \\ 0 \equiv 0 \pmod{20}. \end{gathered}$$

This confirms that $p = 20k+3$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{3even} = \{p \in \mathbb{P} : \;\; p \equiv 3 \pmod{20}\} = \left\{p \in \mathbb{P} : \;\; 2p^{-1}(10^{p-1}-1) \equiv 306 \pmod{360}\right\}. \tag{25}$$

□

(7) **Prime class** P_{7even}.

$$P_{7even} = \{p \in \mathbb{P} : \;\; p \equiv 7 \pmod{20}\} = \left\{p \in \mathbb{P} : \;\; 2p^{-1}(10^{p-1}-1) \equiv 234 \pmod{360}\right\}$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1}-1) \equiv 234 \pmod{360}$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20}$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 7 \pmod{20} \;\Rightarrow\; \frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20}$$

Suppose $p \equiv 7 \pmod{20}$, i.e., $p = 20k+7$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+6}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+6} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k+6} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 7 \equiv 7 \pmod{20}.$$

The multiplicative inverse of 7 in $\mathbb{Z}_{20}$ is a number x, such that:

$$7x \equiv 1 \pmod{20}.$$

We have to,

$$7(3) \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} = 7^{-1} \equiv 3 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1}-1) = 3(19) \equiv 57 \equiv 17 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 9(17) = 153 \equiv 13 \pmod{20}.$$

Then,

$$\frac{1}{9}\,p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20}.$$

Therefore,

$$p \equiv 7 \pmod{20} \;\Rightarrow\; \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20}. \tag{26}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20} \Rightarrow\; p \equiv 7 \pmod{20}$$

Suppose,

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1}-1) \equiv 117 \equiv 17 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1}-1 \equiv 17p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 17p+1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k+7$, for any integer k, is a congruence solution.

$$\begin{gathered} 10^{p-1} = 10^{20k+6} \equiv 17(20k+7)+1 \pmod{20}. \\ 0 \equiv 140k+119+1 \pmod{20}. \\ 0 \equiv 0 \pmod{20}. \end{gathered}$$

This confirms that $p = 20k+7$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{7even} = \{p \in \mathbb{P}: \;\; p \equiv 7 \pmod{20}\} = \left\{p \in \mathbb{P}: \;\; 2p^{-1}(10^{p-1}-1) \equiv 234 \pmod{360}\right\}. \tag{27}$$

□

(8) **Prime class** P_{9even}.

$$P_{9even} = \{p \in \mathbb{P}: \;\; p \equiv 9 \pmod{20}\} = \left\{p \in \mathbb{P}: \;\; 2p^{-1}(10^{p-1}-1) \equiv 342 \pmod{360}\right\}$$

Let's start by simplifying this congruence,

$$2p^{-1}(10^{p-1}-1) \equiv 342 \pmod{360}$$

Dividing both sides by 18, we have:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20}$$

($\Rightarrow$) Now let us prove that:

$$p \equiv 9 \pmod{20} \Rightarrow \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20}$$

Suppose $p \equiv 9 \pmod{20}$, i.e., $p = 20k+9$, for any integer k.
Let's calculate 10^{p-1} module 20. Replacing p, we have to $10^{p-1} = 10^{20k+8}$.
And we know that any power $k \geq 2$ is congruent to 0 module 20, then:

$$10^{20k+8} \equiv 0 \pmod{20}.$$

Then,

$$10^{p-1} - 1 = 10^{20k+8} - 1 \equiv -1 \equiv 19 \pmod{20}.$$

Let's calculate the inverse of p module 20.

$$p = 20k + 9 \equiv 9 \pmod{20}.$$

The multiplicative inverse of 9 in $\mathbb{Z}_{20}$ is a number x, such that:

$$9x \equiv 1 \pmod{20}.$$

We have to,

$$9(9) \equiv 1 \pmod{20}.$$

Therefore, the inverse of p modulo 20 is:

$$p^{-1} = 9^{-1} \equiv 9 \pmod{20}.$$

Let us now substitute in the congruence what we have obtained:

$$p^{-1}(10^{p-1}-1) = 9(19) \equiv 171 \equiv 11 \pmod{20}.$$

We need to calculate 1/9 module 20. The inverse of 9 modulo 20 is 9, because

$$9(9) = 81 \equiv 1 \pmod{20}.$$

Multiplying by 1/9, we have to:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 9(11) = 99 \equiv 19 \pmod{20}.$$

Then,

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20}.$$

Therefore,

$$p \equiv 9 \pmod{20} \Rightarrow \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20}. \tag{28}$$

□

($\Leftarrow$) Now let us prove that:

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20} \Rightarrow p \equiv 9 \pmod{20}$$

Suppose,

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20}$$

Multiplying both sides by 9

$$p^{-1}(10^{p-1}-1) \equiv 171 \equiv 11 \pmod{20}$$

Multiplying both sides by p

$$10^{p-1}-1 \equiv 11p \pmod{20}$$

Reordering,

$$10^{p-1} \equiv 11p+1 \pmod{20}$$

We know that,

$$10^{p-1} \equiv 0 \pmod{20}.$$

Let's see if $p = 20k+9$, for any integer k, is a congruence solution.

$$\begin{gathered} 10^{p-1} = 10^{20k+8} \equiv 11(20k+9)+1 \pmod{20}. \\ 0 \equiv 220k+99+1 \pmod{20}. \\ 0 \equiv 0 \pmod{20}. \end{gathered}$$

This confirms that $p = 20k+9$, is the general solution of the congruence.
Therefore, we conclude that:

$$P_{9even} = \{p \in \mathbb{P} : \quad p \equiv 9 \pmod{20}\} = \left\{p \in \mathbb{P} : \quad 2p^{-1}(10^{p-1}-1) \equiv 342 \pmod{360}\right\}. \tag{29}$$

□

By the above proof, we can write the 8 prime classes in terms of module 20.

Lemma 4.1. *The* 8 *prime classes module* 20 *are:*

(i) ***Prime classes penult-odd***

$$P_{1odd} = \left\{p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 1 \pmod{20} \ \Leftrightarrow \ p \equiv 11 \pmod{20}\right\}.$$

$$P_{3odd} = \left\{p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 3 \pmod{20} \ \Leftrightarrow \ p \equiv 17 \pmod{20}\right\}.$$

$$P_{7odd} = \left\{p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 7 \pmod{20} \ \Leftrightarrow \ p \equiv 13 \pmod{20}\right\}.$$

$$P_{9odd} = \left\{p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 9 \pmod{20} \ \Leftrightarrow \ p \equiv 19 \pmod{20}\right\}.$$

(ii) ***Prime classes penult-even***

$$P_{1even} = \left\{p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 11 \pmod{20} \ \Leftrightarrow \ p \equiv 1 \pmod{20}\right\}.$$

$$P_{3even} = \left\{p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20} \ \Leftrightarrow \ p \equiv 7 \pmod{20}\right\}.$$

$$P_{7even} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 17 \pmod{20} \ \Leftrightarrow \ p \equiv 3 \pmod{20} \right\}.$$

$$P_{9even} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 19 \pmod{20} \ \Leftrightarrow \ p \equiv 9 \pmod{20} \right\}.$$

Remark. Each prime class consists of two congruences module 20. An interesting property is that when adding the residues of each pair of congruences, we obtain 0 module 20 for classes P_{3odd}, P_{3even}, P_{7odd}, P_{7even}. For classes P_{1odd}, P_{1even}, the sum of residues is 12 module 20 and for classes P_{9odd}, P_{9even}, the sum of residuals is 8 module 20. Thus, by adding the new residues of the classes P_{1odd} with P_{9odd} y P_{1even} with P_{9even}, is obtained to be 0 module 20.

Corollary 4.1.1. *The sum of the residues of the congruences of the prime classes module* 20 *is:*

- 0 *for* P_{3odd}, P_{3even}, P_{7odd}, P_{7even}.
- 12 *for* P_{1odd}, P_{1even}.
- 8 *for* P_{9odd}, P_{9even}.

Proof.
(i) Adding the residuals of each pair of congruences, we have:

- $P_{1odd} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 1 \pmod{20} \ \Leftrightarrow \ p \equiv 11 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p \equiv 1 + 11 \equiv 12 \pmod{20}.$$

- $P_{3odd} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 3 \pmod{20} \ \Leftrightarrow \ p \equiv 17 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p \equiv 3 + 17 \equiv 0 \pmod{20}.$$

- $P_{7odd} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 7 \pmod{20} \ \Leftrightarrow \ p \equiv 13 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p \equiv 7 + 13 \equiv 0 \pmod{20}.$$

- $P_{9odd} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 9 \pmod{20} \ \Leftrightarrow \ p \equiv 19 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p \equiv 9 + 19 \equiv 8 \pmod{20}.$$

By adding up the congruences of the classes P_{1odd} y P_{9odd}, we have:

$$\left(\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p\right) + \left(\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p\right) \equiv 12 + 8 \equiv 0 \pmod{20}$$

(ii) Adding the residuals of each pair of congruences, we have:

- $P_{1even} = \left\{ p \in \mathbb{P} : \ \frac{1}{9}\, p^{-1}(10^{p-1} - 1) \equiv 11 \pmod{20} \ \Leftrightarrow \ p \equiv 1 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1} - 1) + p \equiv 11 + 1 \equiv 12 \pmod{20}$$

- $P_{3even} = \left\{ p \in \mathbb{P} : \ \dfrac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 13 \pmod{20} \ \Leftrightarrow \ p \equiv 7 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) + p \equiv 13 + 7 \equiv 0 \pmod{20}$$

- $P_{7even} = \left\{ p \in \mathbb{P} : \ \dfrac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 17 \pmod{20} \ \Leftrightarrow \ p \equiv 3 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) + p \equiv 17 + 3 \equiv 0 \pmod{20}$$

- $P_{9even} = \left\{ p \in \mathbb{P} : \ \dfrac{1}{9}\, p^{-1}(10^{p-1}-1) \equiv 19 \pmod{20} \ \Leftrightarrow \ p \equiv 9 \pmod{20} \right\}.$

$$\frac{1}{9}\, p^{-1}(10^{p-1}-1) + p \equiv 19 + 9 \equiv 8 \pmod{20}$$

When summing the congruences of the classes P_{1even} y P_{9even}, we have:

$$\left(\frac{1}{9}\, p^{-1}(10^{p-1}-1) + p\right) + \left(\frac{1}{9}\, p^{-1}(10^{p-1}-1) + p\right) \equiv 12 + 8 \equiv 0 \pmod{20}.$$

□

5 Golden prime symmetry theorem

Theorem 5.1. *For each prime $p > 5$, if $p^{-1}(10^{p-1}-1)$ is the decimal period of the reciprocals of the prime numbers, then it is defined:*

$$G(p) = 2\sin(2p^{-1}(10^{p-1}-1)) \tag{30}$$

(i) This function $G : \mathbb{P} \to \mathbb{R} : p \longmapsto G(p)$, satisfies:

$$G(p) = 2\sin(2p^{-1}(10^{p-1}-1)) = \frac{\pm 1 \pm \sqrt{5}}{2} \tag{31}$$

Explicitly:

$$G(p) = 2\sin(2p^{-1}(10^{p-1}-1)) = \begin{cases} \varphi & \Leftrightarrow \quad p \in P_{3odd}, P_{7odd}. \\ \varphi^{-1} & \Leftrightarrow \quad p \in P_{1odd}, P_{9odd}. \\ -\varphi & \Leftrightarrow \quad p \in P_{3even}, P_{7even}. \\ -\varphi^{-1} & \Leftrightarrow \quad p \in P_{1even}, P_{9even}. \end{cases} \tag{32}$$

(ii) An equivalence relation is induced in 8 *prime classes module* 360.

$$P_{1odd}, P_{3odd}, P_{7odd}, P_{9odd}, P_{1even}, P_{3even}, P_{7even}, P_{9even}.$$

Proof.

(i) Let us find in general form the decimal period of the reciprocals of the prime numbers $p > 5$. Let

$$\frac{1}{p} = 0.\overline{a_1 a_2 ... a_{k-1} a_k}.$$

Multiplying both sides by 10^{p-1}, we have:

$$\frac{10^{p-1}}{p} = a_1 a_2 ... a_{k-1} a_k.\overline{a_1 a_2 ... a_{k-1} a_k}.$$

Subtracting both sides $1/p$, we have:

$$\frac{10^{p-1}}{p} - \frac{1}{p} = a_1 a_2 ... a_{k-1} a_k.$$

This decimal period $p^{-1}(10^{p-1} - 1)$ has length k, which by Midy's generalized theorem can be even or odd. Moreover, the decimal period of the reciprocals of the prime numbers can be cut into n blocks which when added together give strings of nines.

By Fermat's little theorem [33], Let $a = 10$, $p > 5$ be, then $gcd(10, p) = 1$.

$$a^{p-1} \equiv 1 \pmod{p}.$$

$$10^{p-1} \equiv 1 \pmod{p}.$$

Therefore, $0, 90, 180, 270$ (which are multiples of 5), are not residues of $p^{-1}(10^{p-1}-1) \pmod{360}$, i.e.,

$$p^{-1}(10^{p-1} - 1) \pmod{360} \notin \{0, 90, 180, 270\}.$$

Using the 8 proofs of the prime classes of the section 4, we have:

$$p \pmod{20} \in \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

$$2p^{-1}(10^{p-1} - 1) \pmod{360} \in \{18, 54, 126, 162, 198, 234, 306, 342\}.$$

Therefore, for the table 3.

$$G(p) = 2\sin(2p^{-1}(10^{p-1} - 1)) = \frac{\pm 1 \pm \sqrt{5}}{2}. \tag{33}$$

□

(ii) Let's call

$$\alpha(p) = 2p^{-1}(10^{p-1} - 1). \tag{34}$$

Based on the equations (15),(17),(19),(21),(23),(25),(27),(29), we have 8 prime classes module 360 that appear in the table 5 2.1. These classes are:

$$P_{1odd}, P_{3odd}, P_{7odd}, P_{9odd}, P_{1even}, P_{3even}, P_{7even}, P_{9even}.$$

Now let us prove that the relation is reflexive, symmetric and transitive. We define the relation $p \sim q$ over $\mathbb{P}$.

Reflexive:

Let $p \in \mathbb{P}$ be, if $2p^{-1}(10^{p-1} - 1) \equiv 2p^{-1}(10^{p-1} - 1) \pmod{360}$, then,

$\alpha(p) \equiv 18 \pmod{360}$ $\Leftrightarrow$	$p \in P_{1odd}$	$2\sin(\alpha(p)) = 2\sin(18) = \varphi^{-1}$
$\alpha(p) \equiv 54 \pmod{360}$ $\Leftrightarrow$	$p \in P_{7odd}$	$2\sin(\alpha(p)) = 2\sin(54) = \varphi$
$\alpha(p) \equiv 126 \pmod{360}$ $\Leftrightarrow$	$p \in P_{3odd}$	$2\sin(\alpha(p)) = 2\sin(126) = \varphi$
$\alpha(p) \equiv 162 \pmod{360}$ $\Leftrightarrow$	$p \in P_{9odd}$	$2\sin(\alpha(p)) = 2\sin(162) = \varphi^{-1}$
$\alpha(p) \equiv 198 \pmod{360}$ $\Leftrightarrow$	$p \in P_{1even}$	$2\sin(\alpha(p)) = 2\sin(198) = -\varphi^{-1}$
$\alpha(p) \equiv 234 \pmod{360}$ $\Leftrightarrow$	$p \in P_{7even}$	$2\sin(\alpha(p)) = 2\sin(234) = -\varphi$
$\alpha(p) \equiv 306 \pmod{360}$ $\Leftrightarrow$	$p \in P_{3even}$	$2\sin(\alpha(p)) = 2\sin(306) = -\varphi$
$\alpha(p) \equiv 342 \pmod{360}$ $\Leftrightarrow$	$p \in P_{9even}$	$2\sin(\alpha(p)) = 2\sin(342) = -\varphi^{-1}$

Table 5: The 8 equivalence prime classes.

$$360 \Big| \Big(2p^{-1}(10^{p-1} - 1) - 2p^{-1}(10^{p-1} - 1)\Big) \tag{35}$$

then, $360|0$, i.e., $0 = (360)0$. It follows that $p \sim p$. Therefore, it is reflexive.

Symmetrical:
Let $p, q \in \mathbb{P}$ be, it must be fulfilled that if $2p^{-1}(10^{p-1} - 1) \equiv 2q^{-1}(10^{q-1} - 1) \pmod{360}$, then, $2q^{-1}(10^{q-1} - 1) \equiv 2p^{-1}(10^{p-1} - 1) \pmod{360}$.
By hypothesis,

$$2p^{-1}(10^{p-1} - 1) \equiv 2q^{-1}(10^{q-1} - 1) \pmod{360}$$

i.e.,

$$360 \Big| \Big(2p^{-1}(10^{p-1} - 1) - 2q^{-1}(10^{q-1} - 1)\Big)$$

then,

$$360 \Big| - \Big(2p^{-1}(10^{p-1} - 1) - 2q^{-1}(10^{q-1} - 1)\Big)$$

then,

$$360 \Big| \Big(2q^{-1}(10^{q-1} - 1) - 2p^{-1}(10^{p-1} - 1)\Big)$$

i.e.,

$$2q^{-1}(10^{q-1} - 1) \equiv 2p^{-1}(10^{p-1} - 1) \pmod{360} \tag{36}$$

It follows that, $p \sim q$ y $q \sim p$. Therefore, it is symmetrical.

Transitive:
Let $p, q, r \in \mathbb{P}$ be, it must be fulfilled that if $2p^{-1}(10^{p-1} - 1) \equiv 2q^{-1}(10^{q-1} - 1) \pmod{360}$ and $2q^{-1}(10^{q-1} - 1) \equiv 2r^{-1}(10^{r-1} - 1) \pmod{360}$, then $2p^{-1}(10^{p-1} - 1) \equiv 2r^{-1}(10^{r-1} - 1) \pmod{360}$.
By hypothesis,

$$2p^{-1}(10^{p-1} - 1) \equiv 2q^{-1}(10^{q-1} - 1) \pmod{360}$$

and

$$2q^{-1}(10^{q-1}-1) \equiv 2r^{-1}(10^{r-1}-1) \pmod{360}$$

i.e.,

$$360 \Big| \left(2p^{-1}(10^{p-1}-1) - 2q^{-1}(10^{q-1}-1)\right)$$

and

$$360 \Big| \left(2q^{-1}(10^{q-1}-1) - 2r^{-1}(10^{r-1}-1)\right)$$

then,

$$360 \Big| \left((2p^{-1}(10^{p-1}-1) - 2q^{-1}(10^{q-1}-1)) + (2q^{-1}(10^{q-1}-1) - 2r^{-1}(10^{r-1}-1))\right)$$

then,

$$360 \Big| \left(2p^{-1}(10^{p-1}-1) - 2r^{-1}(10^{r-1}-1)\right)$$

Therefore,

$$2p^{-1}(10^{p-1}-1) \equiv 2r^{-1}(10^{r-1}-1) \pmod{360} \tag{37}$$

It follows that if $p \sim q$ y $q \sim r$, then $p \sim r$. Therefore, it is transitive. □

5.1 Representatives of prime classes

Corollary 5.1.1. *Let $\alpha(p) = 2p^{-1}(10^{p-1}-1)$ be, the prime class representatives are:*

$$\begin{aligned}
[18] &= \{\ p \in P_{1odd} : \ \ \alpha(p) \equiv 18 \pmod{360}\ \}. \\
[54] &= \{\ p \in P_{3odd} : \ \ \alpha(p) \equiv 54 \pmod{360}\ \}. \\
[126] &= \{\ p \in P_{7odd} : \ \ \alpha(p) \equiv 126 \pmod{360}\ \}. \\
[162] &= \{\ p \in P_{9odd} : \ \ \alpha(p) \equiv 162 \pmod{360}\ \}. \\
[198] &= \{\ p \in P_{1even} : \ \ \alpha(p) \equiv 198 \pmod{360}\ \}. \\
[234] &= \{\ p \in P_{3even} : \ \ \alpha(p) \equiv 234 \pmod{360}\ \}. \\
[306] &= \{\ p \in P_{7even} : \ \ \alpha(p) \equiv 306 \pmod{360}\ \}. \\
[342] &= \{\ p \in P_{9even} : \ \ \alpha(p) \equiv 342 \pmod{360}\ \}.
\end{aligned}$$

5.2 Mersenne prime number M52

On October 21, 2024, the program **GIMPS** (Great Internet Mersenne Prime Search) published the largest new prime number known to mankind. It is the Mersenne prime number 52 with a length of $41,024,320$ digits. On its website: https://www.mersenne.org/primes/?press=M136279841

$$M_{52} = 2^{136,279,841} - 1 = 881694327...486871551.$$

This prime number ends in 1 and its penultimate digit is odd. That is to say, $M_{52} \in P_{1odd}$.

Example: By Theorem 5.1, if $p = M_{52}$, we have:

$$G(M_{52}) = 2\sin(2(M_{52})^{-1}(10^{M_{52}-1}-1)).$$

$$G(2^{136279841}-1) = 2\sin(2(2^{136279841}-1)^{-1}(10^{2^{136279841}-2}-1)) = \frac{\sqrt{5}-1}{2}.$$

6 Theorem of powers of 2 and 3

Theorem 6.1. *The golden ratio is the image of the cosine of the product* $2^n\,3^n$*. We define the function* J *for all* $n > 2$*,* $n \in \mathbb{Z}$*. i.e.:*

$J : \mathbb{N} \to \mathbb{R} : n \longmapsto J(n)$*, such that if:*

$$2^n\,3^n \equiv 216 \pmod{360} \quad \Rightarrow \quad J(n) = 2\cos(2^n\,3^n) = 2\cos(216)^\circ = -\varphi.$$

Proof. By induction on n.

Base step.

For $n = 3$, we have: $2^3\,3^3 = 216$. Therefore, it is true for $n = 3$.

Inductive hypothesis.

Suppose the statement is true for $n = k$.

$$2^k\,3^k \equiv 216 \pmod{360}$$

We want to show that it is true for $n = k + 1$.

$$2^{k+1}\,3^{k+1} \equiv 216 \pmod{360}$$

Inductive step.

$$2^{k+1}\,3^{k+1} = 2\,2^k\,3\,3^k = 6\,(2^k\,3^k) \equiv 6\,(216) \equiv 216 \pmod{360}$$

Where we have used the induction hypothesis.

Thus, we have shown that $2^n\,3^n \equiv 216 \pmod{360}$, for all $n > 2$.

Therefore, it is complied with:

$$2^n\,3^n \equiv 216 \quad \pmod{360} \quad \Rightarrow \quad 2\cos(2^n\,3^n)^\circ = 2\cos(216)^\circ = -\varphi. \tag{38}$$

□

Corollary 6.1.1. *Let* $n \in \mathbb{N}$ *be,* $n \geq 2$*. The following is true:*

(1) $2^{n+1}\,3^n \equiv 72 \pmod{360}, \quad \Rightarrow \quad 2\cos(2^{n+1}\,3^n)^\circ = 2\cos(72)^\circ = \varphi^{-1}.$

(2) $2^{n+1}\,3^{n-1} \equiv 144 \pmod{360} \quad \Rightarrow \quad 2\cos(2^{n+1}\,3^{n-1})^\circ = 2\cos(144)^\circ = -\varphi.$

(3) $2^n\,3^{n+1} \equiv 288 \pmod{360} \quad \Rightarrow \quad 2\cos(2^n\,3^{n+1})^\circ = 2\cos(288)^\circ = \varphi^{-1}.$

Proof.

(1) We will use theorem 6.1, which states that: $2^n\,3^n \equiv 216 \pmod{360}$.

Let's prove that if

$$2^{n+1}\,3^n \equiv 72 \pmod{360} \quad \Rightarrow \quad 2\cos(2^{n+1}\,3^n)^\circ = 2\cos(72)^\circ = \varphi^{-1}.$$

We have to:

$$2^{n+1}\,3^n = 2\,(2^n\,3^n)$$

By the theorem 6.1, we have:

$$2\,(2^n\,3^n) \equiv 2(216) = 432 \equiv 72 \pmod{360}$$

Therefore,

$$2^{n+1}\,3^{n} \equiv 72 \pmod{360} \;\Rightarrow\; 2\cos(2^{n+1}\,3^{n})^{\circ} = 2\cos(72)^{\circ} = \varphi^{-1}. \tag{39}$$

□

(2) Let's prove that if

$$2^{n+1}\,3^{n-1} \equiv 144 \pmod{360} \;\Rightarrow\; 2\cos(2^{n+1}\,3^{n-1})^{\circ} = 2\cos(144)^{\circ} = -\varphi.$$

We have to:

$$2^{n+1}\,3^{n-1} = 2\,(1/3)\,(2^{n}\,3^{n}).$$

By the theorem 6.1, we have:

$$2\,(1/3)(2^{n}\,3^{n}) \equiv 2\,(1/3)(216) = 144 \pmod{360}.$$

Therefore

$$2^{n+1}\,3^{n-1} \equiv 144 \pmod{360} \;\Rightarrow\; 2\cos(2^{n+1}\,3^{n-1})^{\circ} = 2\cos(144)^{\circ} = -\varphi. \tag{40}$$

□

(3) Let's prove that if

$$2^{n}\,3^{n+1} \equiv 288 \pmod{360} \;\Rightarrow\; 2\cos(2^{n}\,3^{n+1})^{\circ} = 2\cos(288)^{\circ} = \varphi^{-1}.$$

We have to:

$$2^{n}\,3^{n+1} = 3\,(2^{n}\,3^{n}).$$

By the theorem 6.1, we have:

$$3\,(2^{n}\,3^{n}) \equiv 3(216) = 648 \equiv 288 \pmod{360}.$$

Therefore

$$2^{n}\,3^{n+1} \equiv 288 \pmod{360} \;\Rightarrow\; 2\cos(2^{n}\,3^{n+1})^{\circ} = 2\cos(288)^{\circ} = \varphi^{-1} \tag{41}$$

□

7 Prime golden rotation matrix

Theorem 7.1. *Let the angles be $\sigma = \{18, 54, 126, 162, 198, 234, 306, 342\}$ and let $\alpha(p) = 2p^{-1}(10^{p-1}-1)$ and $p > 5$ prime. The following rotation matrices are equivalent:*

$$\begin{pmatrix} \cos(18)^{\circ} & -\sin(18)^{\circ} \\ \sin(18)^{\circ} & \cos(18)^{\circ} \end{pmatrix} = \begin{pmatrix} \cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p)) \end{pmatrix} \;\Leftrightarrow\; p \in P_{1odd} \tag{42}$$

$$\begin{pmatrix}\cos(54)^\circ & -\sin(54)^\circ \\ \sin(54)^\circ & \cos(54)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{7odd} \tag{43}$$

$$\begin{pmatrix}\cos(126)^\circ & -\sin(126)^\circ \\ \sin(126)^\circ & \cos(126)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{3odd} \tag{44}$$

$$\begin{pmatrix}\cos(162)^\circ & -\sin(162)^\circ \\ \sin(162)^\circ & \cos(162)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{9odd} \tag{45}$$

$$\begin{pmatrix}\cos(198)^\circ & -\sin(198)^\circ \\ \sin(198)^\circ & \cos(198)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{1even} \tag{46}$$

$$\begin{pmatrix}\cos(234)^\circ & -\sin(234)^\circ \\ \sin(234)^\circ & \cos(234)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{7even} \tag{47}$$

$$\begin{pmatrix}\cos(306)^\circ & -\sin(306)^\circ \\ \sin(306)^\circ & \cos(306)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{3even} \tag{48}$$

$$\begin{pmatrix}\cos(342)^\circ & -\sin(342)^\circ \\ \sin(342)^\circ & \cos(342)^\circ\end{pmatrix} = \begin{pmatrix}\cos(\alpha(p)) & -\sin(\alpha(p)) \\ \sin(\alpha(p)) & \cos(\alpha(p))\end{pmatrix} \quad \Leftrightarrow \quad p \in P_{9even} \tag{49}$$

Proof.
By the proofs in section 4, these equalities are satisfied in general for rotation matrices. □

7.1 Eigenvalues of the prime golden rotation matrix

Theorem 7.2. *The characteristic polynomials of each rotation matrix, whose zeros are the eigenvalues of the matrix, correspond to the prime equivalence classes in the complex plane.*

(1) *Characteristic polynomial:* $\lambda^2 - \sqrt{2+\varphi}\lambda + 1 = 0$.

$$P_{1odd}: \quad \lambda_1 = \frac{\sqrt{2+\varphi}}{2} + \frac{\sqrt{2-\varphi}}{2} i = e^{i18^\circ} \tag{50}$$

$$P_{9even}: \quad \lambda_2 = \frac{\sqrt{2+\varphi}}{2} - \frac{\sqrt{2-\varphi}}{2} i = e^{i342^\circ} \tag{51}$$

(2) *Characteristic polynomial:* $\lambda^2 + \sqrt{2+\varphi}\lambda + 1 = 0$.

$$P_{9odd}: \quad \lambda_1 = -\frac{\sqrt{2+\varphi}}{2} + \frac{\sqrt{2-\varphi}}{2} i = e^{i162^\circ} \tag{52}$$

$$P_{1even}: \quad \lambda_2 = -\frac{\sqrt{2+\varphi}}{2} - \frac{\sqrt{2-\varphi}}{2}i = e^{i198^\circ} \tag{53}$$

(3) *Characteristic polynomial:* $\lambda^2 + \sqrt{3-\varphi}\lambda + 1 = 0$.

$$P_{3odd}: \quad \lambda_1 = -\frac{\sqrt{3-\varphi}}{2} + \frac{\varphi}{2}i = e^{i126^\circ} \tag{54}$$

$$P_{7even}: \quad \lambda_2 = -\frac{\sqrt{3-\varphi}}{2} - \frac{\varphi}{2}i = e^{i234^\circ} \tag{55}$$

(4) *Characteristic polynomial:* $\lambda^2 - \sqrt{3-\varphi}\lambda + 1 = 0$.

$$P_{7odd}: \quad \lambda_1 = \frac{\sqrt{3-\varphi}}{2} + \frac{\varphi}{2}i = e^{i54^\circ} \tag{56}$$

$$P_{3even}: \quad \lambda_2 = \frac{\sqrt{3-\varphi}}{2} - \frac{\varphi}{2}i = e^{i306^\circ} \tag{57}$$

8 Fifth roots of prime classes

The polar form of a complex number $z = a + bi$, where $a = r\cos\sigma$ y $b = r\sin\sigma$ replacing each term, we have: $z = re^{i\sigma} = r(\cos\sigma + i\sin\sigma)$.

Theorem 8.1. *The eigenvalues of the rotation matrices correspond to the* 8 *prime equivalence classes in the complex plane. Let* $\alpha(p) = 2p^{-1}(10^{p-1} - 1)$ *be, we have:*

$$z_1 = i^{1/5} = e^{i18^\circ} = \frac{\sqrt{2+\varphi}}{2} + \frac{\sqrt{2-\varphi}}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{1odd}$$

$$z_2 = i^{3/5} = e^{i54^\circ} = \frac{\sqrt{3-\varphi}}{2} + \frac{\varphi}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{7odd}$$

$$z_3 = i^{7/5} = e^{i126^\circ} = -\frac{\sqrt{3-\varphi}}{2} + \frac{\varphi}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{3odd}$$

$$z_4 = i^{9/5} = e^{i162^\circ} = -\frac{\sqrt{2+\varphi}}{2} + \frac{\sqrt{2-\varphi}}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{9odd}$$

$$z_5 = i^{-9/5} = e^{i198^\circ} = -\frac{\sqrt{2+\varphi}}{2} - \frac{\sqrt{2-\varphi}}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{1even}$$

$$z_6 = i^{-7/5} = e^{i234^\circ} = -\frac{\sqrt{3-\varphi}}{2} - \frac{\varphi}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{7even}$$

$$z_7 = i^{-3/5} = e^{i306^\circ} = \frac{\sqrt{3-\varphi}}{2} - \frac{\varphi}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{3even}$$

$$z_8 = i^{-1/5} = e^{i342^\circ} = \frac{\sqrt{2+\varphi}}{2} - \frac{\sqrt{2-\varphi}}{2}i = \cos(\alpha(p)) + i\sin(\alpha(p)) \quad \Leftrightarrow \quad p \in P_{9even}$$

m	1	2	3	4	5	6	7	8	9	10	11	12
$\pi(m)$	1	3	8	6	20	24	16	12	24	60	10	24

Table 6: Pisano periods.

Proof.

As shown in table 5 and the identities in tables 3 and 4, we have the equivalences between notable golden angles that extend to the complex plane. □

In addition, the product of the conjugates is:

$$z_1 \ \ z_8 = e^{i18^\circ} e^{i342^\circ} = 1.$$
$$z_2 \ \ z_7 = e^{i54^\circ} e^{i306^\circ} = 1.$$
$$z_3 \ \ z_6 = e^{i126^\circ} e^{i234^\circ} = 1.$$
$$z_4 \ \ z_5 = e^{i162^\circ} e^{i198^\circ} = 1.$$

9 Fibonacci symmetry theorems

9.1 Pisano period

For any positive integer m, the Fibonacci sequence is periodic module m. The existence of the periodic functions of the Fibonacci numbers was discovered by Joseph Louis Lagrange in 1774 [29].

Definition 9.1. The length of the period of the Fibonacci sequence module m is designated $\pi(m)$ and is called *Pisano period*.

In table 7, we see the lengths of the Pisano periods for the first twelve values of m. In table 6 we see that the Pisano period $\pi(9) = 24$, corresponding to the residues cycle:

$$\{ \ 0, 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1 \ \}.$$

Separating the cycle of 24 residues module 9 in two blocks of 12 elements each and adding the blocks together, we have a string of nines.

$$011235843718 + 088764156281 = 99999999999.$$

9.2 Fibonacci divisibility

For any positive integer k is fulfilled [9]:

$$(1)\ 2|F_{3k} \qquad (2)\ 3|F_{4k} \qquad (3)\ 5|F_{5k} \qquad (4)\ 4|F_{6k} \qquad (5)\ 13|F_{7k} \qquad (6)\ 7|F_{8k}$$
$$(7)\ 17|F_{9k} \qquad (8)\ 11|F_{10k} \qquad (9)\ \{6, 9, 12, 16\}|F_{12k} \qquad (10)\ 29|F_{14k}$$
$$(11)\ \{10, 61\}|F_{15k} \qquad (12)\ 15|F_{20k}.$$

For the following theorem 11.1, we use as domain only the twelfth Fibonacci numbers that are not multiples of 5. This is because multiples of 5 in the Fibonacci sequence occur in the indexes that are multiples of 5. That is, $5|F_{5m}$.

Therefore, the residue module 360 of the six sixtieth Fibonacci numbers does not belong to the set of golden angles shown in the table 1, because they end in 0 or 5. This set is:

$F_1 \equiv 1 \pmod 9$	$F_{13} \equiv 8 \pmod 9$
$F_2 \equiv 1 \pmod 9$	$F_{14} \equiv 8 \pmod 9$
$F_3 \equiv 2 \pmod 9$	$F_{15} \equiv 7 \pmod 9$
$F_4 \equiv 3 \pmod 9$	$F_{16} \equiv 6 \pmod 9$
$F_5 \equiv 5 \pmod 9$	$F_{17} \equiv 4 \pmod 9$
$F_6 \equiv 8 \pmod 9$	$F_{18} \equiv 1 \pmod 9$
$F_7 \equiv 4 \pmod 9$	$F_{19} \equiv 5 \pmod 9$
$F_8 \equiv 3 \pmod 9$	$F_{20} \equiv 6 \pmod 9$
$F_9 \equiv 7 \pmod 9$	$F_{21} \equiv 2 \pmod 9$
$F_{10} \equiv 1 \pmod 9$	$F_{22} \equiv 8 \pmod 9$
$F_{11} \equiv 8 \pmod 9$	$F_{23} \equiv 1 \pmod 9$
$F_{12} \equiv 0 \pmod 9$	$F_{24} \equiv 0 \pmod 9$

Table 7: Pisano period $\pi(9)$.

$$F_{12(5n)} = F_{60n)} = \{F_{60}, F_{120}, F_{180}, F_{240}, ...\},\ n \in \mathbb{Z}^+.$$

Definition 9.2. The twelfth Fibonacci numbers that are not multiples of 5 are of the form: $F_{12(5m+j)}$ with $k = 5m + j$, $m \in \mathbb{Z}$, $j = 1, 2, 3, 4$.

Result: The golden ratio is the image of the discrete space of a subsession of the Fibonacci sequence, namely the twelfth Fibonacci numbers that are not multiples of 5, by a certain function. And module 360 produces a partition into 8 infinite families or Fibonacci classes that are invariant under the isometries of the regular pentagon and its opposite rotation.

10 Equivalence classes of twelfth Fibonacci

Proposition 3. Let $F_{12(5m+j)}$, such that $k = 5m + j$, $m \in \mathbb{Z}$, $j = 1, 2, 3, 4$. The 8 equivalence classes of the twelfth Fibonacci for all $k \in \mathbb{Z}^+$, is fulfilled:

$$\begin{aligned}
F_{12k} &\equiv 144 \pmod{360} \quad \textit{when} \quad k \equiv 1 \pmod 5.\\
F_{12k} &\equiv 288 \pmod{360} \quad \textit{when} \quad k \equiv 2 \pmod 5.\\
F_{12k} &\equiv 72 \pmod{360} \quad \textit{when} \quad k \equiv 3 \pmod 5.\\
F_{12k} &\equiv 216 \pmod{360} \quad \textit{when} \quad k \equiv 4 \pmod 5.
\end{aligned}$$

$$\begin{aligned}
F_{12k} - 180 &\equiv 324 \pmod{360} \quad \textit{when} \quad k \equiv 1 \pmod 5.\\
F_{12k} - 180 &\equiv 108 \pmod{360} \quad \textit{when} \quad k \equiv 2 \pmod 5.\\
F_{12k} - 180 &\equiv 252 \pmod{360} \quad \textit{when} \quad k \equiv 3 \pmod 5.\\
F_{12k} - 180 &\equiv 36 \pmod{360} \quad \textit{when} \quad k \equiv 4 \pmod 5.
\end{aligned}$$

Proof.

Case 1. Let us prove that $F_{12k} \equiv 144 \pmod{360}$ when $k \equiv 1 \pmod 5$.
We know that $k \equiv 1 \pmod 5$, i.e., $k = 5m + 1$, for any integer m.
Replacing k, we have:

$$F_{12(5m+1)} = F_{60m+12}$$

We know that the Fibonacci numbers are periodic and furthermore $5|F_{5m}$. Then,

$$F_{60m+12} \equiv F_{12} \pmod{360}, \text{ because } F_{60m} \equiv F_{12(5m)} \equiv 0 \pmod{360}.$$

and as:

$$F_{12} = 144 \equiv 144 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} \equiv 144 \pmod{360} \quad when \quad k \equiv 1 \pmod{5}. \tag{58}$$

□

Case 2. Let us prove that $F_{12k} \equiv 288 \pmod{360}$ when $k \equiv 2 \pmod{5}$.
We know that $k \equiv 2 \pmod{5}$, i.e., $k = 5m + 2$, for any integer m.
Replacing k, we have:

$$F_{12(5m+2)} = F_{60m+24}$$

then,

$$F_{60m+24} \equiv F_{24} \pmod{360}.$$

and as:

$$F_{24} = 46368 \equiv 288 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} \equiv 288 \pmod{360} \quad when \quad k \equiv 2 \pmod{5}. \tag{59}$$

□

Case 3. Let us prove that $F_{12k} \equiv 72 \pmod{360}$ when $k \equiv 3 \pmod{5}$.
We know that $k \equiv 3 \pmod{5}$, i.e., $k = 5m + 3$, for any integer m.
Replacing k, we have:

$$F_{12(5m+3)} = F_{60m+36}$$

then,

$$F_{60m+36} \equiv F_{36} \pmod{360}.$$

and as:

$$F_{36} = 14930352 \equiv 72 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} \equiv 72 \pmod{360} \quad when \quad k \equiv 3 \pmod{5}. \tag{60}$$

□

Case 4. Let us prove that $F_{12k} \equiv 216 \pmod{360}$ when $k \equiv 4 \pmod 5$.
We know that $k \equiv 4 \pmod 5$, i.e., $k = 5m + 4$, for any integer m.
Replacing k, we have:

$$F_{12(5m+4)} = F_{60m+48}$$

then,

$$F_{60m+48} \equiv F_{48} \pmod{360}.$$

and as:

$$F_{48} = 4807526976 \equiv 216 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} \equiv 216 \pmod{360} \quad when \quad k \equiv 4 \pmod 5 \tag{61}$$

□

Case 5. Let us prove that $F_{12k} - 180 \equiv 324 \pmod{360}$ when $k \equiv 1 \pmod 5$.
We know that $F_{12k} \equiv 144 \pmod{360}$ when $k \equiv 1 \pmod 5$ by the result (57).
Subtracting 180, we have:

$$F_{12k} - 180 \equiv 144 - 180 \equiv -36 \equiv 324 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} - 180 \equiv 324 \pmod{360} \quad when \quad k \equiv 1 \pmod 5 \tag{62}$$

□

Case 6. Let us prove that $F_{12k} - 180 \equiv 108 \pmod{360}$ when $k \equiv 2 \pmod 5$.
We know that $F_{12k} \equiv 288 \pmod{360}$ when $k \equiv 2 \pmod 5$ by the result (58).
Subtracting 180, we have:

$$F_{12k} - 180 \equiv 288 - 180 \equiv 108 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} - 180 \equiv 108 \pmod{360} \quad when \quad k \equiv 2 \pmod 5 \tag{63}$$

□

Case 7. Let us prove that $F_{12k} - 180 \equiv 252 \pmod{360}$ when $k \equiv 3 \pmod 5$.
We know that $F_{12k} \equiv 72 \pmod{360}$ when $k \equiv 3 \pmod 5$ by the result (59).
Subtracting 180, we have:

$$F_{12k} - 180 \equiv 72 - 180 \equiv -108 \equiv 252 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} - 180 \equiv 252 \pmod{360} \quad \textit{when} \quad k \equiv 3 \pmod{5} \tag{64}$$

□

Case 8. Let us prove that $F_{12k} - 180 \equiv 36 \pmod{360}$ when $k \equiv 4 \pmod{5}$. We know that $F_{12k} \equiv 216 \pmod{360}$ when $k \equiv 4 \pmod{5}$ by the result (60). Subtracting 180, we have:

$$F_{12k} - 180 \equiv 216 - 180 \equiv 36 \pmod{360}.$$

Therefore, it is concluded that:

$$F_{12k} - 180 \equiv 36 \pmod{360} \quad \textit{when} \quad k \equiv 4 \pmod{5} \tag{65}$$

□

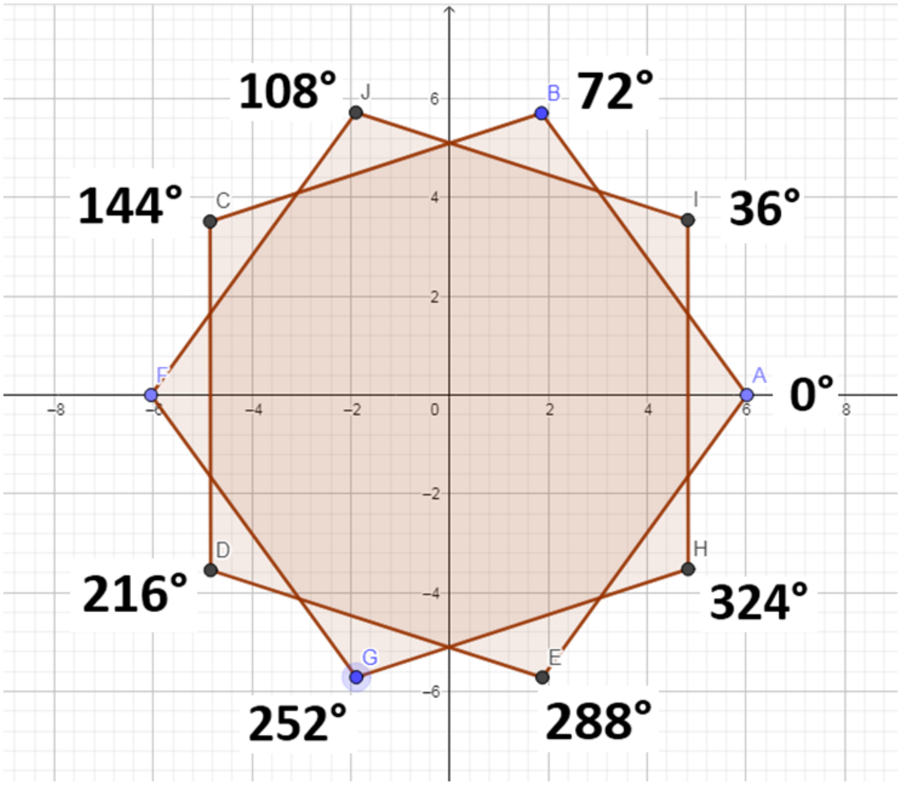

Figure 3: Isometries of the twelfths Fibonacci.

11 Theorem of twelfth Fibonacci

Theorem 11.1. *(i) For every twelfth Fibonacci number of the form* $F_{12(5m+j)}$*, where* $k = 5m + j$*,* $m \in \mathbb{Z}$*,* $j = 1, 2, 3, 4$*,* $\varphi = (1 + \sqrt{5})/2$*, is defined:*

$H : \mathbb{F}_{12k} \to \mathbb{R} : k \longmapsto H(k)$*, satisfies:*

$F_{12(5m+3)} \equiv 72 \pmod{360}$	$2\cos(F_{12(5m+3)}) = 2\cos(72) = \varphi^{-1}$
$F_{12(5m+1)} \equiv 144 \pmod{360}$	$2\cos(F_{12(5m+1)}) = 2\cos(144) = -\varphi$
$F_{12(5m+4)} \equiv 216 \pmod{360}$	$2\cos(F_{12(5m+4)}) = 2\cos(216) = -\varphi$
$F_{12(5m+2)} \equiv 288 \pmod{360}$	$2\cos(F_{12(5m+2)}) = 2\cos(288) = \varphi^{-1}$
$F_{12(5m+4)} - 180 \equiv 36 \pmod{360}$	$2\cos(F_{12(5m+4)} - 180) = 2\cos(36) = \varphi$
$F_{12(5m+2)} - 180 \equiv 108 \pmod{360}$	$2\cos(F_{12(5m+2)} - 180) = 2\cos(108) = -\varphi^{-1}$
$F_{12(5m+3)} - 180 \equiv 252 \pmod{360}$	$2\cos(F_{12(5m+3)} - 180) = 2\cos(252) = -\varphi^{-1}$
$F_{12(5m+1)} - 180 \equiv 324 \pmod{360}$	$2\cos(F_{12(5m+1)} - 180) = 2\cos(324) = \varphi$

Table 8: The 8 Fibonacci classes $F_{12(5m+j)}$.

$$H(k) = 2\cos(F_{12(5m+j)}) = \begin{cases} -\varphi & \Leftrightarrow \quad j = 1, 4. \\ \varphi^{-1} & \Leftrightarrow \quad j = 2, 3. \end{cases} \tag{66}$$

$$H(k) = 2\cos(F_{12(5m+j)} - 180) = \begin{cases} \varphi & \Leftrightarrow \quad j = 1, 4. \\ -\varphi^{-1} & \Leftrightarrow \quad j = 2, 3. \end{cases} \tag{67}$$

(ii) *An equivalence relation is induced on the twelfth Fibonacci numbers* $F_{12(5m+j)}$.

Proof.
(i) We have, by definition of $j = 1, 2, 3, 4$.

$$F_{12(5m+j)} \pmod{360} \notin \{0, 90, 180, 270\}.$$
$$F_{12(5m+j)} - 180 \pmod{360} \notin \{0, 90, 180, 270\}.$$

By the equations (58),(59),(60),(61),(62),(63),(64),(65) we have:

$$F_{12(5m+j)} \pmod{360} \in \{72, 144, 216, 288\}.$$
$$F_{12(5m+j)} - 180 \pmod{360} \in \{36, 108, 252, 324\}.$$

Based on Table 8, there are 8 equivalence classes of Fibonacci twelfths:

$$F_{12(5m+1)}, \quad F_{12(5m+2)}, \quad F_{12(5m+3)}, \quad F_{12(5m+4)}$$
$$F_{12(5m+1)} - 180, \quad F_{12(5m+2)} - 180, \quad F_{12(5m+3)} - 180, \quad F_{12(5m+4)} - 180.$$

(ii) *Let* $a, b \in \mathbb{F}_{12(5m+j)}$ *be,* $m \in \mathbb{Z}$*,* $j = \{1, 2, 3, 4\}$*, is defined,* $a \sim b \Leftrightarrow a \equiv b \equiv \pmod{360}$*.*
Let us prove that the relation is reflexive, symmetric and transitive.

Reflexive: Let $a \in \mathbb{F}_{12(5m+j)}$ be, if $a \equiv b \pmod{360}$, then, $360|(a-b)$, then, $360|0$, i.e., $0 = (360)0$. It follows that $a \sim a$. Therefore, it is reflexive.

Symmetrical: Let $a, b \in \mathbb{F}_{12(5m+j)}$ be, it must be fulfilled that if $a \equiv b \pmod{360}$, then $b \equiv a \pmod{360}$. By hypothesis, $a \equiv b \pmod{360}$, i.e., $360|(a-b)$, then $360|-(a-b)$, then, $360|(b-a)$, then, $b \equiv a \pmod{360}$. It follows that $a \sim b$ y $b \sim a$. Therefore, it is symmetric.

Transitive: Let $a, b, c \in \mathbb{F}_{12(5m+j)}$ be, it must be fulfilled that if $a \equiv b \pmod{360}$ y $b \equiv c \pmod{360}$, then $a \equiv c \pmod{360}$. By hypothesis, $a \equiv b \pmod{360}$ y $b \equiv c \pmod{m}$, i.e., $360|(a-b)$ y $360|(b-c)$, then, $m|((a-b)+(b-c))$. then, $360|(a-c)$. Therefore, $a \equiv c \pmod{360}$. It follows that if $a \sim b$ y $b \sim a$, then $a \sim c$. Therefore, it is transitive. $\square$

11.1 Class representatives of the twelfths Fibonacci

Corollary 11.1.1. *Class representatives of the twelfths Fibonacci $F_{12(5m+j)}$, where $m \in \mathbb{Z}$, $j = 1, 2, 3, 4$, are:*

$$
\begin{aligned}
[72] &= \{\ a \in \mathbb{F}_{12(5m+3)} :\ \ a \equiv 72 \pmod{360}\ \}.\\
[144] &= \{\ a \in \mathbb{F}_{12(5m+1)} :\ \ a \equiv 144 \pmod{360}\ \}.\\
[216] &= \{\ a \in \mathbb{F}_{12(5m+4)} :\ \ a \equiv 216 \pmod{360}\ \}.\\
[288] &= \{\ a \in \mathbb{F}_{12(5m+2)} :\ \ a \equiv 288 \pmod{360}\ \}.\\
[36] &= \{\ a \in \mathbb{F}_{12(5m+4)} - 180 :\ \ a \equiv 36 \pmod{360}\ \}.\\
[108] &= \{\ a \in \mathbb{F}_{12(5m+2)} - 180 :\ \ a \equiv 108 \pmod{360}\ \}.\\
[252] &= \{\ a \in \mathbb{F}_{12(5m+3)} - 180 :\ \ a \equiv 252 \pmod{360}\ \}.\\
[324] &= \{\ a \in \mathbb{F}_{12(5m+1)} - 180 :\ \ a \equiv 324 \pmod{360}\ \}.
\end{aligned}
$$

12 Rotation matrix of twelfth Fibonacci

Theorem 12.1. *Let the angles be $\delta = \{36^\circ, 72^\circ, 108^\circ, 144^\circ, 216^\circ, 252^\circ, 288^\circ, 324^\circ\}$ and $F_{12(5m+j)}$, where $m \in \mathbb{Z}$, $j = 1, 2, 3, 4$. The following rotation matrices are equivalent:*

$$\begin{pmatrix} \cos(36)^\circ & -\sin(36)^\circ \\ \sin(36)^\circ & \cos(36)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+4)} - 180) & -\sin(F_{12(5m+4)} - 180) \\ \sin(F_{12(5m+4)} - 180) & \cos(F_{12(5m+4)} - 180) \end{pmatrix} \tag{68}$$

$$\begin{pmatrix} \cos(72)^\circ & -\sin(72)^\circ \\ \sin(72)^\circ & \cos(72)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+3)}) & -\sin(F_{12(5m+3)}) \\ \sin(F_{12(5m+3)}) & \cos(F_{12(5m+3)}) \end{pmatrix} \tag{69}$$

$$\begin{pmatrix} \cos(108)^\circ & -\sin(108)^\circ \\ \sin(108)^\circ & \cos(108)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+2)} - 180) & -\sin(F_{12(5m+2)} - 180) \\ \sin(F_{12(5m+2)} - 180) & \cos(F_{12(5m+2)} - 180) \end{pmatrix} \tag{70}$$

$$\begin{pmatrix} \cos(144)^\circ & -\sin(144)^\circ \\ \sin(144)^\circ & \cos(144)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+1)}) & -\sin(F_{12(5m+1)}) \\ \sin(F_{12(5m+1)}) & \cos(F_{12(5m+1)}) \end{pmatrix} \tag{71}$$

$$\begin{pmatrix} \cos(216)^\circ & -\sin(216)^\circ \\ \sin(216)^\circ & \cos(216)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+4)}) & -\sin(F_{12(5m+4)}) \\ \sin(F_{12(5m+4)}) & \cos(F_{12(5m+4)}) \end{pmatrix} \tag{72}$$

$$\begin{pmatrix} \cos(252)^\circ & -\sin(252)^\circ \\ \sin(252)^\circ & \cos(252)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+3)} - 180) & -\sin(F_{12(5m+3)} - 180) \\ \sin(F_{12(5m+3)} - 180) & \cos(F_{12(5m+3)} - 180) \end{pmatrix} \tag{73}$$

$$\begin{pmatrix} \cos(288)^\circ & -\sin(288)^\circ \\ \sin(288)^\circ & \cos(288)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+2)}) & -\sin(F_{12(5m+2)}) \\ \sin(F_{12(5m+2)}) & \cos(F_{12(5m+2)}) \end{pmatrix} \tag{74}$$

$$\begin{pmatrix} \cos(324)^\circ & -\sin(324)^\circ \\ \sin(324)^\circ & \cos(324)^\circ \end{pmatrix} = \begin{pmatrix} \cos(F_{12(5m+1)} - 180) & -\sin(F_{12(5m+1)} - 180) \\ \sin(F_{12(5m+1)} - 180) & \cos(F_{12(5m+1)} - 180) \end{pmatrix} \tag{75}$$

Proof.

From the results obtained in table 8, the equality of the angles and the equivalence classes is demonstrated. This implies the equality of the rotation matrices.

13 Eigenvalues of rotation matrix of twelfth Fibonacci

Theorem 13.1. *The characteristic polynomials of each rotation matrix, whose zeros are the eigenvalues of the matrix, correspond to the equivalence classes of the twelfth Fibonacci numbers in the complex plane.*

(1) *Characteristic polynomial:* $\lambda^2 - \varphi\lambda + 1 = 0.$

$$F_{12(5m+4)} - 180: \quad \lambda_1 = \frac{\varphi}{2} + \frac{\sqrt{3-\varphi}}{2}i = e^{i36^\circ} \tag{76}$$

$$F_{12(5m+1)} - 180: \quad \lambda_2 = \frac{\varphi}{2} - \frac{\sqrt{3-\varphi}}{2}i = e^{i324^\circ} \tag{77}$$

(2) *Characteristic polynomial:* $\lambda^2 - \varphi^{-1}\lambda + 1 = 0.$

$$F_{12(5m+3)}: \quad \lambda_1 = \frac{\sqrt{2-\varphi}}{2} + \frac{\sqrt{2+\varphi}}{2}i = e^{i72^\circ} \tag{78}$$

$$F_{12(5m+2)} - 180: \quad \lambda_2 = \frac{\sqrt{2-\varphi}}{2} - \frac{\sqrt{2+\varphi}}{2}i = e^{i288^\circ} \tag{79}$$

(3) *Characteristic polynomial:* $\lambda^2 + \varphi\lambda + 1 = 0.$

$$F_{12(5m+1)}: \quad \lambda_1 = -\frac{\varphi}{2} + \frac{\sqrt{3-\varphi}}{2}i = e^{i144^\circ} \tag{80}$$

$$F_{12(5m+4)}: \quad \lambda_2 = -\frac{\varphi}{2} - \frac{\sqrt{3-\varphi}}{2}i = e^{i216^\circ} \tag{81}$$

(4) *Characteristic polynomial:* $\lambda^2 + \varphi^{-1}\lambda + 1 = 0.$

$$F_{12(5m+2)} - 180: \quad \lambda_1 = -\frac{\sqrt{2-\varphi}}{2} + \frac{\sqrt{2+\varphi}}{2}i = e^{i108^\circ} \tag{82}$$

$$F_{12(5m+3)} - 180: \quad \lambda_2 = -\frac{\sqrt{2-\varphi}}{2} - \frac{\sqrt{2+\varphi}}{2}i = e^{i252^\circ} \tag{83}$$

14 Fifth roots of Fibonacci classes

Theorem 14.1. *The eigenvalues of the rotation matrices correspond to the* 8 *equivalence classes of the twelfth Fibonacci* $F_{12(5m+j)}$*, where* $m \in \mathbb{Z}$*,* $j = 1, 2, 3, 4.$ *in the complex plane.*

P_n	Prime classes	F_{12k}	Fibonacci classes
P_{1odd}	$i^{1/5} = e^{\pi i/10} = e^{i18^\circ}$	$F_{12(5m+4)} - 180$	$i^{2/5} = e^{\pi i/5} = e^{i36^\circ}$
P_{7odd}	$i^{3/5} = e^{3\pi i/10} = e^{i54^\circ}$	$F_{12(5m+3)}$	$i^{4/5} = e^{2\pi i/5} = e^{i72^\circ}$
P_{3odd}	$i^{7/5} = e^{7\pi i/10} = e^{i126^\circ}$	$F_{12(5m+2)} - 180$	$i^{6/5} = e^{3\pi i/5} = e^{i108^\circ}$
P_{9odd}	$i^{9/5} = e^{9\pi i/10} = e^{i162^\circ}$	$F_{12(5m+1)}$	$i^{8/5} = e^{4\pi i/5} = e^{i144^\circ}$
P_{1even}	$i^{-9/5} = e^{-9\pi i/10} = e^{i198^\circ}$	$F_{12(5m+4)}$	$i^{-8/5} = e^{-4\pi i/5} = e^{i216^\circ}$
P_{7even}	$i^{-7/5} = e^{-7\pi i/10} = e^{i234^\circ}$	$F_{12(5m+3)} - 180$	$i^{-6/5} = e^{-3\pi i/5} = e^{i252^\circ}$
P_{3even}	$i^{-3/5} = e^{-3\pi i/10} = e^{i306^\circ}$	$F_{12(5m+2)}$	$i^{-4/5} = e^{-2\pi i/5} = e^{i288^\circ}$
P_{9even}	$i^{-1/5} = e^{-\pi i/10} = e^{i342^\circ}$	$F_{12(5m+1)} - 180$	$i^{-2/5} = e^{-\pi i/5} = e^{i324^\circ}$

Table 9: Prime classes and Fibonacci classes.

$$z_1 = i^{2/5} = e^{i36^\circ} = \frac{\varphi}{2} + \frac{\sqrt{3-\varphi}}{2}i = \cos(F_{12(5m+4)} - 180) + i\sin(F_{12(5m+4)} - 180).$$

$$z_2 = i^{4/5} = e^{i72^\circ} = \frac{\sqrt{2-\varphi}}{2} + \frac{\sqrt{2+\varphi}}{2}i = \cos(F_{12(5m+3)}) + i\sin(F_{12(5m+3)}).$$

$$z_3 = i^{6/5} = e^{i108^\circ} = -\frac{\sqrt{2-\varphi}}{2} + \frac{\sqrt{2+\varphi}}{2}i = \cos(F_{12(5m+2)} - 180) + i\sin(F_{12(5m+2)} - 180).$$

$$z_4 = i^{8/5} = e^{i144^\circ} = -\frac{\varphi}{2} + \frac{\sqrt{3-\varphi}}{2}i = \cos(F_{12(5m+1)}) + i\sin(F_{12(5m+1)}).$$

$$z_5 = i^{-8/5} = e^{i216^\circ} = -\frac{\varphi}{2} - \frac{\sqrt{3-\varphi}}{2}i = \cos(F_{12(5m+4)}) + i\sin(F_{12(5m+4)}).$$

$$z_6 = i^{-6/5} = e^{i252^\circ} = -\frac{\sqrt{2-\varphi}}{2} - \frac{\sqrt{2+\varphi}}{2}i = \cos(F_{12(5m+3)} - 180) + i\sin(F_{12(5m+3)} - 180).$$

$$z_7 = i^{-4/5} = e^{i288^\circ} = \frac{\sqrt{2-\varphi}}{2} - \frac{\sqrt{2+\varphi}}{2}i = \cos(F_{12(5m+2)}) + i\sin(F_{12(5m+2)}).$$

$$z_8 = i^{-2/5} = e^{i324^\circ} = \frac{\varphi}{2} - \frac{\sqrt{3-\varphi}}{2}i = \cos(F_{12(5m+1)} - 180) + i\sin(F_{12(5m+1)} - 180).$$

Proof.

From table 3 and 4 we have the equivalences between notable golden angles extending to the complex plane. □

In addition, the product of the conjugates is:

$$z_1 \;\; z_8 = e^{i36^\circ} e^{i324^\circ} = 1.$$
$$z_2 \;\; z_7 = e^{i72^\circ} e^{i288^\circ} = 1.$$
$$z_3 \;\; z_6 = e^{i108^\circ} e^{i252^\circ} = 1.$$
$$z_4 \;\; z_5 = e^{i144^\circ} e^{i216^\circ} = 1.$$

15 Invertible elements in $\mathbb{Z}_{20}$

Theorem 15.1. *The prime equivalence classes module* 20 *correspond to the invertible elements in* $\mathbb{Z}_{20}$.

$$\{1, 3, 7, 9, 11, 13, 17, 19\}.$$

$(i^{1/5})^2 = (e^{\pi i/10})^2 = e^{\pi i/5} = e^{i36^\circ}$
$(i^{3/5})^2 = (e^{3\pi i/10})^2 = e^{3\pi i/5} = e^{i108^\circ}$
$(i^{7/5})^2 = (e^{7\pi i/10})^2 = e^{-3\pi i/5} = e^{i252^\circ}$
$(i^{9/5})^2 = (e^{9\pi i/10})^2 = e^{-\pi i/5} = e^{i324^\circ}$
$(i^{-9/5})^2 = (e^{-9\pi i/10})^2 = e^{\pi i/5} = e^{i36^\circ}$
$(i^{-7/5})^2 = (e^{-7\pi i/10})^2 = e^{3\pi i/5} = e^{i108^\circ}$
$(i^{-3/5})^2 = (e^{-3\pi i/10})^2 = e^{-3\pi i/5} = e^{i252^\circ}$
$(i^{-1/5})^2 = (e^{-\pi i/10})^2 = e^{-\pi i/5} = e^{i324^\circ}$

Table 10: Square powers of prime classes equal Fibonacci inverse classes.

$gcd(1, 20) = 1$	$gcd(11, 20) = 1$
$gcd(3, 20) = 1$	$gcd(13, 20) = 1$
$gcd(7, 20) = 1$	$gcd(17, 20) = 1$
$gcd(9, 20) = 1$	$gcd(19, 20) = 1$

Table 11: Table of invertible elements in $\mathbb{Z}_{20}$.

Proof.
A number a in $\mathbb{Z}_{20}$ is invertible if there exists a number b, such that:

$$a\,b \equiv 1 \pmod{20}.$$

This happens, if and only if, a y 20 are coprimes, i.e., $gcd(a, 20) = 1$. We know that $20 = 2^2\,5$. To find the numbers in $\mathbb{Z}_{20}$ which are relative primes with 20, we need to find all the numbers in the range of 0 a 19, that are divisible neither by 2 nor by 5. That is, numbers that do not have 2 or 5 as prime factors.

Based on table 11, the invertible elements in $\mathbb{Z}_{20}$, are:

$$\mathbb{Z}_{20}^* = \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

The multiplicative group of units $\mathbb{Z}_{20}^*$, is the set of invertible elements in $\mathbb{Z}_{20}$ under the multiplication operation module 20.

Therefore, the 8 prime equivalence classes module 20 correspond to the invertible elements in $\mathbb{Z}_{20}$.

$$\{1, 3, 7, 9, 11, 13, 17, 19\}.$$

□

15.1 Analysis of $\mathbb{Z}_{20}^*$

- **Order of the group:** The group $\mathbb{Z}_{20}^*$ has 8 elements.
- **Abelian:** The group $\mathbb{Z}_{20}^*$ is finite abelian, because the multiplication modulo 20 is commutative, i.e., for any $a, b \in \mathbb{Z}_{20}^*$, se cumple que: $ab \equiv ba \pmod{20}$.
- **Non-cyclic:** No element of $\mathbb{Z}_{20}^*$ can generate all the elements of the group through their powers.

*	1	3	7	9	11	13	17	19
1	1	3	7	9	11	13	17	19
3	3	9	1	7	13	19	11	17
7	7	1	9	3	17	11	19	13
9	9	7	3	1	19	17	13	11
11	11	13	17	19	1	3	7	9
13	13	19	11	17	3	9	1	7
17	17	11	19	13	7	1	9	3
19	19	17	13	11	9	7	3	1

Table 12: Group of elements of $\mathbb{Z}_{20}^*$.

The 3 y el 7 generate a subgroup $\{1,3,7,9\}$ of order 4, which contains half of the elements of $\mathbb{Z}_{20}^*$. As we will see below:

$$3^1 \equiv 3 \pmod{20}, \quad 3^2 \equiv 9 \pmod{20}, \quad 3^3 \equiv 7 \pmod{20}, \quad 3^4 \equiv 1 \pmod{20}.$$

$$7^1 \equiv 7 \pmod{20}, \quad 7^2 \equiv 9 \pmod{20}, \quad 7^3 \equiv 3 \pmod{20}, \quad 7^4 \equiv 1 \pmod{20}.$$

The 9 only generates a subgroup $\{1,9\}$ de orden 2. Does not generate $\{3,7\}$.

$$9^1 \equiv 9 \pmod{20}, \quad 9^2 \equiv 1 \pmod{20}, \quad 9^3 \equiv 9 \pmod{20}, \quad 9^4 \equiv 1 \pmod{20}.$$

15.2 Structure of $\mathbb{Z}_{20}^*$

According to finite group theory, this group has the structure of a direct product of cyclic groups, and is isomorphic to:

$$\mathbb{Z}_{20}^* \cong \mathbb{Z}_2 X \mathbb{Z}_4.$$

But this structure refers to the sub-groups generated by the elements 3 (of order 4) y 9 (of order 2) in $\mathbb{Z}_{20}^*$.

Under the operation of powers, the structure of the $\mathbb{Z}_{20}^*$ behaves as follows:

- **Elements of order** 4: As 3, their powers run through the elements $\{1,3,7,9\}$.
- **Elements of order** 2: As 9, their powers run through the elements $\{1,9\}$.

These subgroups are not cyclic under the power operation, since they do not generate to all $\mathbb{Z}_{20}^*$.

Conditions for the direct product to be cyclic

The direct product of 2 cyclic groups G y H, denoted GXH, is cyclic, if and only if, the orders of G y H are relative primes, i.e,

$$gcd(|G|,|H|) = 1.$$

Where $|G|$ y $|H|$ are the orders of the groups G y H.

For direct product $\mathbb{Z}_2 X \mathbb{Z}_4$, we have:

$$gcd(|\mathbb{Z}_2|,|\mathbb{Z}_4|) = gcd(2,4) = 2.$$

Therefore, $\mathbb{Z}_{20}^* = \mathbb{Z}_2 X \mathbb{Z}_4$ is not cyclic, i.e., there is no single generator.

16 Cyclotomic polynomials

The ancient Greek geometers posed the problem of dividing a circumference into equal parts using only ruler and compass. This problem is known today as **cyclotomy**. If we have a circle in the complex plane with center at the origin and radius 1, dividing it into n equal parts, it is satisfied that each of the points that cut it is a solution of the polynomial $z^n = 1$, where $z \in \mathbb{C}$, $n \in \mathbb{N}$, $n > 1$. This polynomial is defined as **cyclotomic polynomial** of order n and is denoted as Φ_n, such that it is a unit polynomial whose roots are all the primitive roots of order n of unity [2,3,4,30].

Definition 16.1. Let be the roots of the unit in $\mathbb{C}$ given by:

$$\zeta_k := e^{2\pi ik/n}, \quad 0 \leq k < n, \quad gcd(k, n) = 1, \quad k \in \mathbb{Z}$$

The nth cyclotomic polynomial is defined as follows:

$$\Phi_n(x) = \prod_{k=1}^{n} (x - \zeta_k).$$

In this work we obtained 3 cyclotomic polynomials, whose zeros are the 8 prime classes and the 8 classes of the twelfths Fibonacci.

The golden cyclotomic polynomials are:

$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1.$$

$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1.$$

$$\Phi_{20}(x) = x^8 - x^6 + x^4 - x^2 + 1.$$

16.1 Cyclotomic golden prime polynomial

Theorem 16.1. *The* 8 *complex prime equivalence classes correspond to the zeros of cyclotomic polynomial number*20*:*

$$\Phi_{20}(x) = x^8 - x^6 + x^4 - x^2 + 1.$$

The zeros of the cyclotomic polynomial number 20 *are:*

$$\{e^{\pi i/10}, e^{3\pi i/10}, e^{7\pi i/10}, e^{9\pi i/10}, e^{-\pi i/10}, e^{-3\pi i/10}, e^{-7\pi i/10}, e^{-9\pi i/10}\}.$$

Proof.

The cyclotomic polynomial number 20, denoted as $\Phi_{20}(x)$, is the polynomial whose zeros are the primitive roots 20-th of the unit.

To find the zeros of the cyclotomic polynomial $\Phi_{20}(x)$, we need to find the primitive 20-th roots of unity. These roots are the complex numbers that satisfy the equation: $x^{20} = 1$.

The values of $k < 20$ which are relative primes with 20, that is, $gcd(k, 20) = 1$, are:

$$gcd(k, 20) = 1 \quad \text{for } k = \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

Then,

$$x = e^{2\pi ik/20} \quad \text{for } k = \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

Therefore, based on the equivalences of table 2 for radians, the zeros of the cyclotomic polynomial $\Phi_{20}(x)$ are:

$$\{e^{\pi i/10}, e^{3\pi i/10}, e^{7\pi i/10}, e^{9\pi i/10}, e^{-\pi i/10}, e^{-3\pi i/10}, e^{-7\pi i/10}, e^{-9\pi i/10}\}.$$

And from Table 9, we know that these zeros correspond to the 8 prime equivalence classes in the complex plane and can be written as:

$$\{i^{1/5}, i^{3/5}, i^{7/5}, i^{9/5}, i^{-1/5}, i^{-3/5}, i^{-7/5}, i^{-9/5}\}.$$

$\square$

Proposition 4. There is a new identity for the number π, from the cyclotomic polynomial whose zeros are the 8 prime classes.

$$\pi = (\varphi + \varphi^{-1}) \int_0^\infty \frac{dx}{x^8 - x^6 + x^4 - x^2 + 1}.$$

16.2 Cyclotomic polynomials of the twelfth Fibonacci

Theorem 16.2. *The* 8 *complex equivalence classes of the twelfth Fibonacci numbers correspond to the zeros of the cyclotomic polynomials* 5 *y* 10*:*

$$(i) \quad \Phi_5(x) = x^4 + x^3 + x^2 + x + 1.$$

$$(ii) \quad \Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1.$$

The zeros of cyclotomic polynomial number 5 *y* 10 *are:*

$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1 = (x - i^{4/5})(x - i^{-4/5})(x - i^{8/5})(x - i^{-8/5}).$$

$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1 = (x - i^{2/5})(x - i^{-2/5})(x - i^{6/5})(x - i^{-6/5}).$$

Proof.

(i) The cyclotomic polynomial number 5, denoted as $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$, is the polynomial whose zeros are the primitive 5-th roots of the unit.

To find the zeros of the cyclotomic polynomial $\Phi_5(x)$, we need to find the primitive 5-th roots of unity. These roots are the complex numbers that satisfy the equation: $x^5 = 1$.

The values of $k < 5$ which are relative primes with 5, that is, $gcd(k, 5) = 1$, are:

$$gcd(k, 5) = 1 \qquad \text{for } k = \{1, 2, 3, 4\}.$$

Then,

$$x = e^{2\pi ik/5} \quad \text{for } k = \{1, 2, 3, 4\}.$$

Therefore, based on the equivalences of table 2 for radians, the zeros of the cyclotomic polynomial $\Phi_5(x)$ are:

$$\{e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}\} = \{e^{2\pi i/5}, e^{4\pi i/5}, e^{-4\pi i/5}, e^{-2\pi i/5}\}.$$

And from table 9, we know that these zeros correspond to the 8 equivalence classes of the twelfths Fibonacci in the complex plane and can be written as:

$$\{i^{4/5}, i^{8/5}, i^{-8/5}, i^{-4/5}\}.$$

$\square$

(ii) The cyclotomic polynomial number 10, denoted as $\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$, is the polynomial whose zeros are the primitive 10-th roots of unity.

To find the zeros of the cyclotomic polynomial $\Phi_{10}(x)$, we need to find the primitive 5-th roots of unity. These roots are the complex numbers that satisfy the equation: $x^{10} = 1$.

The values of $k < 10$ which are relative primes with 10, that is, $gcd(k, 10) = 1$, are:

$$gcd(k, 10) = 1 \quad \text{for } k = \{1, 3, 7, 9\}.$$

Then,

$$x = e^{2\pi ik/10} \quad \text{for } k = \{1, 3, 7, 9\}.$$

Therefore, based on the equivalences of table 2 for radians, the zeros of the cyclotomic polynomial $\Phi_{10}(x)$ are:

$$\{e^{\pi i/5}, e^{3\pi i/5}, e^{7\pi i/5}, e^{9\pi i/5}\} = \{e^{\pi i/5}, e^{3\pi i/5}, e^{-3\pi i/5}, e^{-\pi i/5}\}.$$

And from table 9, we know that these zeros correspond to the 8 equivalence classes of the twelfths Fibonacci in the complex plane and can be written as:

$$\{i^{2/5}, i^{6/5}, i^{-6/5}, i^{-2/5}\}.$$

$\square$

ID ORCID: https://orcid.org/0009-0008-7125-352X

References

[1] Apostol, Tom. (1976). *Introduction to Analytic Number Theory.* Springer-Verlag, New York Inc. ISBN 0-387-90163-9.

[2] Beshenov, Alexey, *Teoría de números algebraicos*, CIMAT, 2020, p.66.

[3] Bloom, D.M. *On the coefficients of the cyclotomic polynomials.* Amer. Math. Monthly, 75:372–377, 1968. MR0227086 (37:2671)

[4] Brookfield, G. *The Coefficients of Cyclotomic Polynomials.*vol.89, No.3, June 2016. https://www.calstatela.edu/sites/default/files/cyclotomic.pdf

[5] Caldwell, Chris. “The PrimePages" From Web Resource: https://t5k.org/

[6] Chandra, Pravin and Weisstein, Eric W. “Fibonacci Number." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/FibonacciNumber.html

[7] Dilcher, Karl. “The Fibonacci Logos" From Web Resource. https://www.mathstat.dal.ca/fibonacci/index.html

[8] Euclides. *Elementos*, libro V, Editorial Gredos, Madrid, 1991.

[9] Gandhi, K. R. (2011). *Divisibility properties of Fibonacci numbers*, South Asian J Math, pp. 140-144.

[10] Ginsberg, B. (2004). Midy’s (nearly) secret theorem - an extension after 165 years. College Mathematics Journal, 35(1): 26-30.

[11] Gray, Alexander J. (2000). *Digital roots and reciprocals of primes*, Mathematical Gazette 84.09, 86 pp.

[12] Gupta Ankit and Sury B. (2005). *Decimal expansion of* $1/p$ *and subgroup sums*. Integers: Electronic Journal of Combinatorial Number Theory, Vol. 5, A19.

[13] Hardy, G., Wright, E. (2008). An Introduction to the Theory of Numbers. Oxford University Press, Great Clarendon Street, Oxford, 6th edition.

[14] Hoggat, Verner. (1969). *“Fibonacci and Lucas numbers"*. The Fibonacci association. University of Santa Clara.

[15] Kak, Subhash, Chatterjee A. (1981). *On decimal sequences*. IEEE Transactions on Information Theory, vol. IT-27, 647-652 pp.

[16] Kalman, Dan (1996). *Fractions with Cycling Digit Patterns*.The College Mathematics Journal, Vol. 27, No. 2. 109-115 pp.

[17] Knott, Ron. “The Golden section ratio: Phi." From Web Resource: https://r-knott.surrey.ac.uk/Fibonacci/phi.html

[18] Lang Serge, Algebra, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR1878556 (2003e:00003).

[19] Lewittes Joseph (2007). *Midy’s theorem for periodical decimals*. Electronic Journal of Combinatorial Number Theory, Vol. 7.

[20] Martin, H. (2007). Generalizations of Midy’s theorem on repeating decimals. Integers: Electronic Journal of Combinatorial Number Theory, 7(1):19–25.

[21] Michon, Gérard. (2006). Numericana’s Biographies. From Web Resource: http://www.numericana.com/fame/bio.htm#midy

[22] Plouffe, Simon. “The first 1001 Fibonacci Numbers" From Web Resource: http://www.fullbooks.com/The-first-1001-Fibonacci-Numbers.html

[23] Rademacher, H. and Toeplitz, O. *“The enjoyment of mathematics: Selections from mathematics for the amateur"*, Princeton, NJ: Princeton University Press, 1957, p. 158-160.

[24] Sloane, Neil. OEIS Foundation Inc. “The prime numbers". Entry A000040 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000040

[25] Sloane, Neil. OEIS Foundation Inc. “a(n) = Fibonacci(n) mod 9". Entry A007887 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A007887

[26] Sloane, Neil. OEIS Foundation Inc. “Pisano periods (or Pisano numbers): period of Fibonacci numbers mod n". Entry A001175 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A001175

[27] Tapp, Kristopher. (2012). “Symmetry: A Mathematical Exploration". University of Philadelphia, USA. Springer.

[28] Weisstein, Eric W. “Golden Ratio". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/GoldenRatio.html

[29] Weisstein, Eric W. “Pisano Period". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/PisanoPeriod.html

[30] Weisstein, Eric W. “Cyclotomic Polynomial". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/CyclotomicPolynomial.html

[31] Weisstein, Eric W. “de Moivre’s Identity". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/deMoivresIdentity.html

[32] Weisstein, Eric W. “Pentagram". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Pentagram.html

[33] Weisstein, Eric W. “Fermat’s Little Theorem". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/FermatsLittleTheorem.html

[34] Weisstein, Eric W. “Rotation Matrix". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/RotationMatrix.html

[35] Weisstein, Eric W. “Isometry.". From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Isometry.html

[36] Zaldívar, Felipe. (2014) “Introducción a la teoría de números".